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MUKAI LATTICES:
KNOWN STRUCTURES AND OPEN QUESTIONS

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Plan

1. Basic definitions
2. Algebraic Geometric Source
3. Working example: projective spaces
4. Braid group action and mutations
5. Picard – Lefschetz aspects (after V. Golyshev)
6. Diophantine aspects

Mukai lattice is a free $\mathbb{Z}$-module of finite rank $M \cong \mathbb{Z}^n$ equipped with (maybe neither symmetric nor anti-symmetric) unimodular bilinear form

$$M \times M \to \mathbb{Z}, \quad v, w \mapsto \langle v, w \rangle .$$

Unimodularity means that the polar mapping $M \to M^* \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$

$$v \mapsto (u \mapsto \langle u, v \rangle)$$

is an isomorphism of abelian groups.

A basis $e_1, e_2, \dots, e_n$ of M over $\mathbb{Z}$ is called exceptional (or semiorthogonal) if its Gram matrix $\chi_{ij} = \langle e_i, e_j \rangle$ is upper uni-triangular, i.e.

$$\langle e_i, e_j \rangle = 0 \quad \text{for all } i < j$$
$$\langle e_i, e_i \rangle = 1 \quad \text{for all } i$$

Of course, a Mukai lattice M not necessary admits an exceptional basis and not any $v \in M$ with $\langle v, v \rangle = 1$ is included in an exceptional basis.

Euler's form on Grothendieck's group

The Grothendieck group $K_0(X)$ of an algebraic variety X is equipped with the Euler form $\chi(E,F) = \sum_\nu (-1)^\nu \dim \operatorname{Ext}^\nu(E,F)$.

For smooth X and locally free E, F it can be computed by Riemann – Roch:

$$\chi(E,F) = \chi(E^* \otimes F) = \int_X \operatorname{ch}(E^* \otimes F) \cdot \operatorname{td}(T_X).$$

If $K_0(X)$ is a lattice of finite rank, it is a Mukai lattice. If $\mathcal{D}^b(X)$ admits an exceptional basis (i.e. a collection of objects $E_1, E_2, \dots, E_m$ such that

$$\operatorname{Hom}^{\cdot}_{\mathcal{D}^b(X)}\left(E_i, E_j\right) = \begin{cases} \mathbb{C} & \text{for } i = j \\ 0 & \text{for } i > j \end{cases}.$$

and any object of $\mathcal{D}^b(X)$ can be achieved by taking cones of morphisms starting from finite direct sums of E_i's), then the classes $e_i = [E_i]$ in $K_0(X)$ form an exceptional basis of the Mukai lattice $K_0(X)$.

Example: $K_0(\mathbb{P}_n)$

Mukai lattice $M = K_0(\mathbb{P}_n)$ can be canonically identified with the module of integer valued polynomials of degree $\leqslant n$ with rational coefficients:

$$M \xrightarrow{\sim} \{h \in \mathbb{Q}[t] \mid h(\mathbb{Z}) \subset \mathbb{Z} \ \&\ \deg h \leqslant n\}.$$

The isomorphism takes a class $[E] \in K_0(\mathbb{P}_n)$ to the Hilbert polynomial

$$h_E(k) = \langle \mathcal{O}(-k), E \rangle = \chi(E(k))$$

and sends the basis formed by the structure sheaves of projective subspaces

$$\mathcal{O}_{\mathbb{P}_n}, \mathcal{O}_{\mathbb{P}_{n-1}}, \dots, \mathcal{O}_{\mathbb{P}_1}, \mathcal{O}_{\mathbb{P}_0}$$

to the basis formed by binomial coefficients $\gamma_0 = h_{\mathcal{O}_{\mathbb{P}_0}} \equiv 1$ and

$$\gamma_k(t) = h_{\mathcal{O}_{\mathbb{P}_k}}(t) = \frac{1}{k!}(t+1)(t+2)\cdots(t+k), \quad 1 \leqslant k \leqslant n.$$

If we put $D = d/dt$, then the twisting $T : E \mapsto E(1) = E \otimes \mathcal{O}(1)$ goes to shift:

$$T = e^D : h_E(t) \longmapsto h_E(t+1).$$

The restriction onto a hyperplane: $1 - T^{-1} : E \mapsto E|_{\mathbb{P}_{n-1}}$ goes to

$$\nabla = 1 - e^{-D} : h_E(t) \longmapsto h_E(t) - h_E(t-1)$$

To write Riemann – Roch, it is convenient to present Hilbert polynomials as

$$h_F = F(D)\gamma_n ,$$

where $F(D) \in \mathbb{Q}[[D]]$ is a power series in $D = d/dt$. In this terms

$$h_{F^*} = F(-D)\,\gamma_n$$
$$h_{E\otimes F} = E(D) \cdot F(D)\,\gamma_n$$
$$\langle E , F \rangle = E(-D)F(D)\,\gamma_n\,(0).$$

By the Beilinson theorem, any $(n+1)$ consequent invertible sheaves $\mathcal{O}(i)$, say

$$\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(n)$$

form an exceptional basis of $\mathcal{D}^b(\mathbb{P}_n)$.

Their classes $\gamma_n(t+i)$ form an exceptional basis of Mukai lattice $M = K_0(\mathbb{P}_n)$ with upper uni-triangular Gram matrix whose i's diagonal is filled by $\binom{n+i}{i}$.

For $n = 2, 3$ this Matrix looks like

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 4 & 15 & 20 \\ 0 & 1 & 4 & 15 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Braid group action and mutations

The braid group B_n of n threads acts on the set of exceptional bases of rank n Mukai lattice M. Inverse generators g_i, g_i^{-1}, which braid i-th and $(i+1)$-th threads, replace a pair of consequent vectors e_i, e_{i+1} of each exceptional basis by pairs

$$e_{i+1} - \langle e_i, e_{i+1} \rangle \cdot e_i, \, e_i \quad \text{and} \quad e_{i+1}, \, e_i - \langle e_i, e_{i+1} \rangle \cdot e_{i+1}$$

and preserve all the other basic vectors. The generating relations of B_n

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for all i and } g_i g_j = g_j g_i \text{ for } |i-j| > 1$$

are verified by straightforward computation.

More generally, for any $f \in M$ and $e \in M$ such that $\langle e, e \rangle = 1$ we call

$$L_e f \stackrel{\text{def}}{=} f - \langle e, f \rangle e \quad \text{и} \quad R_e f \stackrel{\text{def}}{=} f - \langle f, e \rangle e$$

respectively left mutation and right mutation of f by means of e.

The Serre Operator

The Serre Operator is a linear mapping $\varkappa : M \to M$ defined by prescription

$$\langle u, w \rangle = \langle w, \varkappa u \rangle \quad \forall u, w \in M .$$

In any basis of M the Gram matrix χ and the matrix of $\varkappa$ are related as

$$\varkappa = \chi^{-1} \chi^{t} .$$

The action of $\varkappa$ on the elements of an exceptional basis $e_1, e_2, \dots, e_n$ is the composition of $n-1$ consequent left mutations along an infinite sequence of vectors $(e_k)_{k \in \mathbb{Z}}$ defined by recursive formulae

$$e_{i-n} = \varkappa(e_i) = L_{e_{i-n+1}} \circ \dots \circ L_{e_{i-2}} \circ L_{e_{i-1}} e_i$$
$$e_{i+n} = \varkappa^{-1}(e_i) = R_{e_{i+n-1}} \circ \dots \circ R_{e_{i+2}} \circ R_{e_{i+1}} e_i .$$

Such infinite sequence is called a helix.

Example

The simplest helix in $M = K_0(\mathbb{P}_n)$ consists of invertible sheaves $\mathcal{O}(i)$, $i \in \mathbb{Z}$. The consequent right mutations of $\mathcal{O}$ along $\mathcal{O}(1), \dots, \mathcal{O}(n)$ are

$$R_{\mathcal{O}(k)} \circ \dots \circ R_{\mathcal{O}(2)} \circ R_{\mathcal{O}(1)} \mathcal{O} = \Lambda^k \mathcal{T}(k)$$

(k'th exterior power of the tangent sheaf $\mathcal{T}$ to $\mathbb{P}_n$). Indeed, k'th exterior power of the (twisted) Euler exact triple $0 \to \mathcal{O} \to V \otimes \mathcal{O}(1) \to \mathcal{T}(1) \to 0$ looks like

$$0 \to \Lambda^{k-1}\mathcal{T}(k-1) \to \Lambda^k V \otimes \mathcal{O}(k) \to \Lambda^k \mathcal{T}(k) \to 0 .$$

Since $\Lambda^k V = \mathrm{Hom}(\mathcal{O}, \Lambda^k \mathcal{T}) = \mathrm{Hom}(\Lambda^{k-1}\mathcal{T}, \mathcal{O}(1))^*$, we get in $K_0(\mathbb{P}_n)$:

$$\dim \Lambda^k V = \langle [\mathcal{O}(k)], [\Lambda^k \mathcal{T}(k)] \rangle = \langle [\Lambda^{k-1}\mathcal{T}(k-1)], [\mathcal{O}(k)] \rangle$$
$$[\Lambda^k \mathcal{T}(k)] = \langle [\Lambda^{k-1}\mathcal{T}(k-1)], [\mathcal{O}(k)] \rangle \cdot [\mathcal{O}(k)] - [\Lambda^{k-1}\mathcal{T}(k-1)] .$$

Remark

If the anticanonical divisor $-K_X \subset X$ is ample, then the adjunction exact triple

$$0 \to E \otimes \omega_X \to E \to E|_{-K_X} \to 0$$

shows that the operator

$$\mathrm{Id} - (\cdot \otimes \omega_X) : [E] \mapsto [E] - [E \otimes \omega_X]$$

coincides with the restriction onto the anticanonical divisor $-K_X$:

$$E \mapsto E|_{-K_X},$$

which is nilpontent. Thus, on a smooth Fano variety X the Serre operator on the Mukai lattice $M = K_0(X)$, which takes

$$[E] \mapsto (-1)^{\dim X}[E \otimes \omega_X],$$

is quasi-unipotent with eigenvalues ± 1.

Degression: non-symmetric bilinear forms over $\mathbb{C}$

Let $W = \mathbb{C} \otimes M$. Then the prescription

$$\chi \mapsto \varkappa(\chi) \stackrel{\text{def}}{=} \chi^{-1} \chi^{t}$$

defines $\mathrm{GL}(W)$ equivariant mapping of the non-degenerated bilinear forms on W to the linear automorphisms of W.

Moreover, two bilinear forms are $\mathrm{GL}(W)$-equivalent iff the corresponding Serre operators are conjugated.

The Jordan normal forms for the Serre operators of non-degenerated bilinear forms can be explicitly described. This leads to the following classification of non-degenerated bilinear forms on W.

We say that W is decomposable if $W = V_1 \oplus V_2$, where $V_1 \neq 0$, $V_2 \neq 0$ and

$$\langle V_1, V_2 \rangle = \langle V_2, V_1 \rangle = 0.$$

Then W is a bi-orthogonal direct sum of indecomposable spaces.

Indecomposable spaces with non-degenerated bilinear forms (over $\mathbb{C}$) are:

- $2k$-dimensional space $W_k(\lambda)$ with the Gram matrix $\begin{pmatrix} 0 & I \\ I_\lambda & 0 \end{pmatrix}$ constructed from $k \times k$ blocks

$$I = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \quad \text{and} \quad I_\lambda = \begin{pmatrix} 0 & & & \lambda \\ & & \lambda & 1 \\ & \iddots & \iddots & \\ \lambda & 1 & & 0 \end{pmatrix},$$

  operator $\varkappa$ has two Jordan chains of length k with eigenvalues λ, λ^{-1}

- n-dimensional space U_n with the Gram matrix $\begin{pmatrix} & & & & 1 \\ & 0 & & -1 & 1 \\ & & 1 & -1 & \\ & \iddots & 1 & 0 & \\ \iddots & \iddots & & & \end{pmatrix}$,

  operator $\varkappa$ has one Jordan chain of length n with eigenvalue $(-1)^{n-1}$.

Conjecture 1

Assume that M is a Mukai lattice of type U_n that admits an exceptional basis. Then the group spanned by mutations of exceptional bases and the isometric automorphisms of M acts transitively on set of exceptional bases of M.

This conjecture was verified for $K_0(\mathbb{P}_2)$ by Drezet and Le Potier in 1980's and for $K_0(\mathbb{P}_3)$ by Nogin in 1990's. Nogin's arguments also allow to verify more strong

Conjecture 2

For each $k = 1, 2, \dots, \mathrm{rk} M$ the group from the conjecture 1 acts transitively on the set of all exceptional collections of length k extendible to exceptional bases of M

Local systems

Let $U = \mathbb{CP}_1 \smallsetminus \{x_0, x_1, \dots, x_n\}$ and $\gamma_0, \gamma_1, \dots, \gamma_n \in \pi_1(U)$ be some fixed basic loops about the points x_ν drawn from a fixed base point $p \in U$.

Rank r local system on U is a locally trivial complex rank r vector bundle $\mathscr{L}$ on U equipped with a flat $\mathrm{GL}_r(\mathbb{C})$-connection or, equivalently, an isomorphism class of a representation

$$\varphi : \pi_1(U) \to \mathrm{GL}_r(\mathbb{C})$$

provided by the holonomy of the connection. The latter is given by a collection of $n+1$ linear operators $\varphi_i = \varphi(\gamma_\nu) \in \mathrm{GL}_r(\mathbb{C})$ satisfying

$$\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi(\gamma_n) = 1$$

and considered up to simultaneous conjugation by the same matrix.

Rigid local systems

Representation $\varphi : \pi_1(U) \to \mathrm{GL}_r(\mathbb{C})$ is called rigid, if for any other representation $\varphi' : \pi_1(U) \to \mathrm{GL}_r(\mathbb{C})$ an existence of a collection $g_\nu \in \mathrm{GL}_r(\mathbb{C})$ such that

$$\forall \nu \quad \varphi'(\gamma_\nu) = g_\nu \varphi(\gamma_\nu) g_\nu^{-1}$$

implies an existence of some $g \in \mathrm{GL}_r(\mathbb{C})$ such that

$$\forall \gamma \in \pi_1(U) \quad \varphi'(\gamma) = g\varphi(\gamma)g^{-1} .$$

Rigid representation φ is called Katz local system if all operators $\varphi(\gamma_\nu)$ are quasi-unipotent, i.e. $\varphi = \zeta + \eta$, where η is nilpotent and ζ is semisimple with $\zeta^m = 1$ for some $m \in \mathbb{N}$.

Pull-back $\mathscr{L}' = p^* \mathscr{L}$ of a Katz local system $\mathscr{L}$ along a non-ramified covering $p : U' \twoheadrightarrow U$ is called quasi Katz local system. N. Katz has shown that such the systems are characterized as those realised by means of the Gauss – Manin connection in the middle homologies of pencils of algebraic varieties.

Claim

For each pair of coprime complex monic polynomials of the form

$$p(t) = 1 + p_1 t + \cdots + p_r t^r$$
$$q(t) = 1 + q_1 t + \cdots + q_r t^r$$

there exists a unique irreducible rigid local system φ of rank $r = \deg p = \deg q$ over $U = \mathbb{P}_1 \{ 0, 1, \infty \}$ whose local monodromies $T = \varphi(\gamma_0)$, $S = \varphi(\gamma_1)$ satisfy the conditions

- S is a quasi-reflection, i.e. $\operatorname{rk}(1 - S) = 1$
- eigenvector of S with eigenvalue -1 is cyclic for T, i.e.

  $$\operatorname{im}(1-S), T(\operatorname{im}(1-S)), \dots, T^{r-1}(\operatorname{im}(1-S))$$

  span $\mathbb{C}^r$
- $\det(1 - tT) = q(t)$
- $\det(1 - tST) = p(t)$

Quasi Katz local systems from helices

The Serre operator of Mukai lattice M of type U_{n+1} has the form

$$\varkappa = (-1)^n \mathrm{Id} + \eta\,,$$

where $\quad \eta^{n+1} = 0 \quad$ but $\quad \mathrm{im}\,\eta^n = w_0 \neq 0\,.\quad$ We put

$$\varepsilon = (-1)^{n+1}$$

and consider on $W = M \otimes \mathbb{C}$ a (skew) symmetric form

$$(v,w)_\varepsilon = \langle\, v\,,\, w\,\rangle + \varepsilon\langle\, w\,,\, v\,\rangle = \varepsilon\langle\, v\,,\, \eta w\,\rangle\,.$$

It has 1-dimensional kernel $\mathbb{C}\cdot w_0$. We put

$$V = W\,/\,\mathbb{C}\cdot w_0$$

and write $(*,*)$ for the non-degenerated form on V induced by $(*,*)_\varepsilon$. Given $e \in W$ with $(e,e) = 1$, we write $\sigma_e(v) \overset{\mathrm{def}}{=} v - (e,v)\cdot e \;\; (mod\;\; w_0)\,.$

It follows from the previous ▸ claim that each exceptional basis $e_0, e_1, \dots, e_n \in M$ produces a local system $\sigma : \pi_1(U) \to \mathrm{GL}(V)$ with fibre V on $U = \mathbb{P}_1 \setminus \{x_0, x_1, \dots, x_n, \infty\}$, where

$$x_k = \exp \frac{2\pi i k}{n+1}, \quad 0 \leqslant k \leqslant n,$$

by prescriptions

$$\sigma(\gamma_k) = \sigma_{e_k} : v \mapsto v - (e_\nu, v) \cdot e_\nu \quad \text{for } 0 \leqslant k \leqslant n$$
$$\sigma(\gamma_\infty) = (\sigma(\gamma_0) \circ \sigma(\gamma_1) \circ \cdots \circ \sigma(\gamma_n))^{-1}.$$

In 2000's V. Golyshev has shown that the local system provided by this way from the basis

$$\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \in K_0(\mathbb{P}_n)$$

is quasi Katz.

Rank 3 Mukai lattice of type U_3

J.N.F. of the Serre operator $\varkappa$ on rank 3 Mukai lattice M is

$$\text{either} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that correspond to the types U_3 , $U_2 \oplus U_1$, and $U_1 \oplus W_2(\lambda)$ with $\lambda \neq \pm 1$ distinguished by $\operatorname{tr}(\kappa) = 3, -1, 1 + \lambda + \lambda^{-1}$.

If M admits an exceptional basis with Gram matrix

$$\chi = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} .$$

then $\operatorname{tr}(\kappa) = \operatorname{tr}(\chi^{-1}\chi^t) = 3 - a^2 - b^2 - c^2 + abc$. Thus, M has type U_3 iff $\{a, b, c\}$ satisfy tripled Markov equation $a^2 + b^2 + c^2 = abc$.

Markov's forms

Let $q(x,y) \in \mathbb{Z}[x,y]$ be homogeneous quadratic form of positive discriminant $-\det q > 0$. Write

$$\mu(q) = \min_{(x,y)\in\mathbb{Z}^2 \smallsetminus (0,0)} \left| q(x,y) \right| \cdot \left| \det(q) \right|^{-1/2}$$

for its homogeneous minimum over non-zero elements of $\mathbb{Z}^2 \subset \mathbb{R}^2$.

In 1890's A. Markov has shown that there is precisely one orbit $\mathrm{SL}_2(\mathbb{Z}) \cdot q_1$ such that $\mu_1 = \mu(q_1) = \max_q \mu(q)$. Outside this orbit there is precisely one orbit $\mathrm{SL}_2(\mathbb{Z}) \cdot q_2$ such that $\mu_2 = \mu(q_2) = \max_{q \nsim q_1} \mu(q)$ (and $\mu_2 < \mu_1$) e.t.c.

The decreasing sequence of the maximal homogeneous minima $\mu_1, \mu_2, \dots$ is called the Markov spectrum and the corresponding forms q_i (up to the action of $\mathrm{SL}_2(\mathbb{Z})$) are called the Markov forms.

Markov's equation

Up to permutations and simultaneous change of signs of any two entries, all solutions $\{a,b,c\}$ of the tripled Markov equation $a^2+b^2+c^2=abc$ are produced from $\{3,3,3\}$ by mutations that change one of a,b,c via Vieta:

$$a\mapsto bc-a\,;\quad b\mapsto ac-b\,;\quad c\mapsto ab-c$$

when two other remain fixed. If we change an exceptional basis e_0,e_1,e_2 by

$$\langle e_0,e_1\rangle\cdot e_0-e_1,\,e_0,\,e_2\,;\quad e_0,\,\langle e_1,e_2\rangle\cdot e_1-e_2,\,e_1\,;\quad e_0,\,e_2,\,\langle e_1,e_2\rangle\cdot e_2-e_1$$

respectively, the Gram matrix $\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}$ turns to

$$\begin{pmatrix}1&a&ab-c\\0&1&b\\0&0&1\end{pmatrix},\quad\begin{pmatrix}1&ac-b&a\\0&1&c\\0&0&1\end{pmatrix},\quad\begin{pmatrix}1&b&bc-a\\0&1&c\\0&0&1\end{pmatrix}$$

Thus, each exceptional basis of rank 3 Mukai lattice of type U_3 is achieved from a basis with the Gram matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad (\star)$$

by mutations and changing signs of basic vectors.

Since $(\star)$ coincides with the Gram matrix of the exceptional basis $\mathcal{O}, \mathcal{T}(-1), \mathcal{O}(1)$ in $K_0(\mathbb{P}_2)$, we conclude that $M = K_0(\mathbb{P}_2)$ is a unique (up to isometry) rank 3 Mukai lattice of type U_3 that has an exceptional basis.

Vectors $e \in M$ that can be included in some exceptional basis stay in bijection with the Markov forms. Namely, quadratic form $q(v) = \langle v, v \rangle$ on $e^\perp \subset M$ written in global coordinates $x = \mathrm{rk}(v)$, $y = c_1(v)$ on $K_0(\mathbb{P}_2)$ is a Markov form and all the Markov forms are obtained in this way.

Under this identification the $\mathrm{SL}_2(\mathbb{Z})$ action on the Markov forms turns to one spanned by the dualization $v \mapsto v^*$ and twisting $v \mapsto Tv$.

Davenport forms

The same situation is expected for totally real homogeneous cubic forms $q(x,y,z) \in \mathbb{Q}[x,y,z]$. If

$$q(x,y,z) = \prod_{i=1}^{3} (\alpha_i x + \beta_i y + \gamma_i z) \quad \text{in} \quad \mathbb{R}[x,y,z],$$

then Davenport has shown in 1940-th that the homogeneous minimum

$$\mu(q) = \min_{\mathbb{Z}^3 \smallsetminus 0} q(x,y,z) \cdot \det^{-1} \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$$

achieves its maximum $\mu = 1/7$ along exactly one $\mathrm{SL}_3(\mathbb{Z})$-orbit and outside this orbit the next maximal $\mu = 1/9$ is achieved along exactly one $\mathrm{SL}_3(\mathbb{Z})$-orbit as well.

Nothing more is known after Davenport.

The first two forms of expected Davenports chain are the norms of cubic extensions of $\mathbb{Q}$ spanned by trigonometric irrationalities whose minimal polynomials are

$$t^3 + t^2 - 2t - 1 \qquad \text{with } \mu = 1/7$$
$$t^3 - 3t - 1 \qquad \text{with } \mu = 1/9$$

Open questions

Are these cubic polynomials connected with the Hilbert polynomials of the exceptional sheaves on $\mathbb{P}_3$?

Is the Davenport chain governed by the Mukai lattice $K_0(\mathbb{P}_3)$ like it was for Markov's chain?

THANKS FOR YOUR ATTENTION !