

Set 1. Projective spaces.

- AG1◊1.** Let the ground field \mathbb{k} consist of q elements. Find the total number of k -dimensional **a)** vector subspaces in \mathbb{k}^n **b)** affine subspaces in \mathbb{A}^n **c)** projective subspaces in \mathbb{P}^n . (Hint: to begin with, take $k = 0, 1, 2, \dots$)
- AG1◊2.** Compute the limits of the previous answers as $q \rightarrow 1$.
- AG1◊3.** Find a geometric condition on 3 lines ℓ_1, ℓ_2, ℓ_3 in $\mathbb{P}_2 = \mathbb{P}(V)$ necessary and sufficient for existence a coordinate system in V such that each ℓ_i becomes the infinite line for the standard chart $U_i = U_{x_i}$ in these coordinates.
- AG1◊4.** Given a line ℓ and a point $p \notin \ell$, is it possible to draw the line parallel to ℓ and passing through p using only the ruler?
- AG1◊5.** There are two points on a wall and a ruler whose length is significantly shorter than a distance between the points. Draw a straight line joining the points.
- AG1◊6.** A point P and two non-parallel lines are drawn on a sheet of paper. The intersection point Q of the lines is far outside the sheet border. Using only the ruler, draw a part of line PQ laying inside the sheet.
- AG1◊7 (the Pappus theorem).** Let points a_1, b_1, c_1 be collinear and points a_2, b_2, c_2 be collinear as well. Show that intersection points $(a_1b_2) \cap (a_2b_1)$, $(b_1c_2) \cap (b_2c_1)$, $(c_1a_2) \cap (c_2a_1)$ are collinear too.
- AG1◊8.** Formulate and prove the dual statement¹ to the Pappus theorem.
- AG1◊9 (1st theorem of Desargus).** Given 2 triangles $A_1B_1C_1$ and $A_2B_2C_2$ on \mathbb{P}_2 , show that three intersection points $(A_1B_1) \cap (A_2B_2)$, $(B_1C_1) \cap (B_2C_2)$, $(C_1A_1) \cap (C_2A_2)$ are collinear iff three lines (A_1A_2) , (B_1B_2) , (C_1C_2) are intersecting at one point².
- Hint. For $\mathbb{k} = \mathbb{R}$ simplify the configuration by moving 3 intersection points to infinity, then use the Euclidean geometry. For an arbitrary \mathbb{k} investigate the fixed point set of the linear projective automorphism sending A_1, B_1, C_1 to A_2, B_2, C_2 and preserving the intersection point $(A_1A_2) \cap (B_1B_2) \cap (C_1C_2)$.
- AG1◊10 (2nd theorem of Desargus).** Let a line ℓ pass through three distinct points p, q, r but do not contain any of three other distinct points a, b, c . Show that lines (ap) , (bq) , (cr) are intersecting at one point iff there exists an involution of ℓ that exchanges p, q, r with intersection points of ℓ with lines (bc) , (ca) , (ab) respectively.

Set 2. Conics and quadrics.

- AG1◊11.** Put real Euclidean plane \mathbb{R}^2 into $\mathbb{C}\mathbb{P}_2$ as the real part of the standard chart $U_0 = \mathbb{C}^2$.
- a)** Find two points $A_{\pm} \in \mathbb{C}\mathbb{P}_2$ laying on all conics visible in \mathbb{R}^2 as the circles.
- b)** Let a conic $C \subset \mathbb{C}\mathbb{P}_2$ have at least 3 non-collinear points in \mathbb{R}^2 and pass through A_{\pm} . Show that $C \cap \mathbb{R}^2$ is a circle.
- AG1◊12.** Given 5 lines without triple intersections on \mathbb{P}_2 , how many conics do touch them all?
- AG1◊13.** Consider a circle C in the real euclidean plane \mathbb{R}^2 and write D for a disc bounded by C . Using ruler and compasses, draw a polar line to a given point $p \in D$ and find a pole of a given line ℓ that does not intersect C . (All the polarities are w.r.t. C .)
- AG1◊14.** Using only the ruler, draw a line passing through a given point p and touching a given conic C . Consider two cases: **a)** $p \notin C$ **b)** $p \in C$.
- AG1◊15.** Line (pq) intersects conic C in points r, s . Assuming that all 4 points p, q, r, s are distinct, show that p lies on the polar line of q w.r.t. C iff $\{p, q\}$ are harmonic to $\{r, s\}$ (i.e. $[p, q; r, s] = -1$).

¹that holds in the dual space $\mathbb{P}_2^{\times} = \mathbb{P}(V^*)$ and duals with the annihilators of all the subspaces from the original statement

²pair of triangles with these properties is called *perspective*

AG1♦16. Given 4 mutually skew³ lines in 3D-space, how many lines does intersect them all? Consider the cases when 3D-space in question is: **a)** $\mathbb{C}P_3$ **b)** $\mathbb{R}P_3$ **c)** affine \mathbb{C}^3 **d)** affine \mathbb{R}^3 . Find all possible answers and indicate those which are stable w.r.t. small perturbation of the 4 given lines.

AG1♦17. How many solutions have equations **a)** $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ **b)** $x_1^2 + x_2^2 + x_3^2 = -1$ over the field \mathbb{F}_9 , which consist of 9 elements $a + b\sqrt{-1}$, $a, b = -1, 0, 1$, added and multiplied modulo 3.

Honorary problems

AG1♦18*. Using only the ruler, draw a triangle inscribed in a given smooth conic Q and with sides a, b, c passing through 3 given points A, B, C . How many solutions may have this problem?

Hint. Start 'naive' drawing from any $p \in Q$ and denote by $\gamma(p)$ the return point after passing through A, B, C . Is the mapping $p \mapsto \gamma(p)$ a homography?

AG1♦19*. Formulate and solve projectively dual problem to the previous one.

AG1♦20* (**Rational normal curve**). Verify that the following curves $C \subset \mathbb{P}_d$ can be moved isomorphically to each other by appropriate linear projective automorphisms of \mathbb{P}_d .

a) Write U for the space of linear forms $\alpha_0 t_0 + \alpha_1 t_1$ in (t_0, t_1) and use $(\alpha_0 : \alpha_1)$ as a homogeneous coordinate on $\mathbb{P}_1 = \mathbb{P}(U)$. Also, consider the space $S^d U$, of homogeneous forms in (t_0, t_1) of degree d , write these forms as $\sum_{n=0}^d \binom{d}{n} a_n t_0^n t_1^{d-n}$, where $\binom{d}{n}$ are the binomial coefficients, and use $(a_0 : a_1 : \dots : a_d)$ as homogeneous coordinates on $\mathbb{P}_d = \mathbb{P}(S^d U)$. Then C is the image of the Veronese map $c_d : \mathbb{P}(U) \rightarrow \mathbb{P}(S^d U)$, which takes $\psi \mapsto \psi^d$.

b) In the notations from (a), $C \subset \mathbb{P}(S^d U)$ is given by the condition $\text{rk} \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{d-1} \\ a_1 & a_2 & a_3 & \dots & a_d \end{pmatrix} = 1$.

c) In the notations from (a), C is an image of any map $\mathbb{P}(U) \rightarrow \mathbb{P}(S^d U)$ given in homogeneous coordinates by a rule $t = (\alpha_0 : \alpha_1) \mapsto (f_0(\alpha) : f_1(\alpha) : \dots : f_d(\alpha))$, where f_0, f_1, \dots, f_d is any collection of linearly independent homogeneous polynomials in $\alpha = (\alpha_0, \alpha_1)$ of degree d .

d) Pick up a collection of $(d + 1)$ distinct points $p_0, p_1, \dots, p_d \in \mathbb{P}_1 = \mathbb{P}(\mathbb{k}^2)$, $p_\nu = (\alpha_\nu : \beta_\nu)$. Then C is the image of mapping $\varphi_{p_0, p_1, \dots, p_d} : \mathbb{P}_1 \rightarrow \mathbb{P}_d$ that takes

$$x = (x_0 : x_1) \mapsto (1/\det(p_0, x) : 1/\det(p_1, x) : \dots : 1/\det(p_d, x)) ,$$

where $\det(p_\nu, x) \stackrel{\text{def}}{=} \alpha_\nu x_1 - \beta_\nu x_0$.

e) Pick up any collection of $(d + 3)$ distinct points $p_1, p_2, \dots, p_n, a, b, c \in \mathbb{P}_n$ such that no $(n + 1)$ of them lie in a shared hyperplane and write $\ell_i \simeq \mathbb{P}_1$ for a pencil of hyperplanes passing through all points p_ν except for p_i . Points a, b, c provide the lines ℓ_ν with compatible homographies

$$\psi_{ij} : \ell_j \xrightarrow{\simeq} \ell_i$$

sending 3 hyperplanes passing through a, b, c from the pencil ℓ_j to the similar 3 hyperplanes of ℓ_i . Then the curve C is drawn by the intersection point of d corresponding to each other hyperplanes of all the pencils: $C = \bigcup_{H \in \ell_1} H \cap \psi_{21}(H) \cap \dots \cap \psi_{n1}(H)$.

³in projective space this means «non-intersecting», in affine space this means «not laying in a shared plane»