Set 1. Tensors.

AG2⋄1. Let $\dim V = 3$. Write $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$ for a variety of singular conics in $\mathbb{P}_2 = \mathbb{P}(V)$. Show that

a) S is an algebraic hypersurface (and find the degree of S)

b) point $C \in S$ is a smooth point of S iff the corresponding conic $C \subset \mathbb{P}_n$ is a pair of crossing lines

c) tangent space $T_C S \subset \mathbb{P}_5$ at a smooth point $C \in S$ consists of all conics in $\mathbb{P}_2$ passing through the singular point $\ell_1 \cap \ell_2$ of $C = \ell_1 \cup \ell_2$.

AG2⋄2 (Aronhold's principle). For a finite dimensional vector space V over a field of zero characteristic show that perfect n th tensor powers $v^{\otimes n} = v \otimes v \otimes \cdots \otimes v$, where $v \in V$, span the subspace of all symmetric tensors $\operatorname{Sym}^n(V) \subset V^{\otimes n}$ and explicitly represent symmetric tensor $u \otimes w \otimes w + w \otimes u \otimes w + w \otimes w \otimes u$, where $u, w \in V$ are non-proportional, as a linear combination of proper tensor cubes.

AG2⋄3 (spinor decomposition). Let $V = \operatorname{Hom}(U_-, U_+)$, where $\dim U_\pm = 2$. Show that canonical direct sum decomposition of $V \otimes V$ into symmetric and skew symmetric parts looks like

$$\underbrace{\Big(\big(S^2U_-^* \otimes S^2U_+\big) \oplus \big(\Lambda^2U_-^* \otimes \Lambda^2U_+\big) \Big)}_{S^2V} \bigoplus \underbrace{\Big(\big(S^2U_-^* \otimes \Lambda^2U_+\big) \oplus \big(\Lambda^2U_-^* \otimes S^2U_+\big) \Big)}_{\Lambda^2V} .$$

AG2⋄4. For vector spaces U, V of finite dimensions construct canonical isomorphisms

$$\operatorname{Hom}\big(\, U \otimes \operatorname{Hom}(U,W)\,,\; W\big) \simeq \operatorname{End}\big(\operatorname{Hom}(U,W)\big) \simeq \operatorname{Hom}\big(\, U\,,\; W \otimes \operatorname{Hom}(U,W)^*\big)$$

and describe an element of $\operatorname{End}(\operatorname{Hom}(U,W))$ corresponding to a mapping $U \otimes \operatorname{Hom}(U,W) \to W$ that takes $u \otimes \varphi \mapsto \varphi(u)$.

AG2⋄5. For vector spaces U, V, W of finite dimensions construct canonical isomorphism

$$\operatorname{End}(U \otimes V \otimes W) \simeq \operatorname{Hom}\big(\operatorname{Hom}(U,V) \otimes \operatorname{Hom}(V,W)\,,\; \operatorname{Hom}(U,W)\big)$$

and describe a linear map $\operatorname{Hom}(U,V) \otimes \operatorname{Hom}(V,W) \to \operatorname{Hom}(U,W)$ corresponding to the identity endomorphism $\operatorname{Id} \in \operatorname{End}(U \otimes V \otimes W)$.

AG2⋄6. Let $G = V(g) \subset \mathbb{P}_3 = \mathbb{P}(V)$ be a smooth quadric. Define a bilinear form $\Lambda^2\widetilde{g}$ on Λ^2V by prescription

$$\Lambda^2\widetilde{g}(\, v_1 \wedge v_2\,,\; w_1 \wedge w_2\,) \stackrel{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1,w_1) & \widetilde{g}(v_1,w_2) \\ \widetilde{g}(v_2,w_1) & \widetilde{g}(v_2,w_2) \end{pmatrix} ,$$

a) Show that $\Lambda^2\widetilde{g}$ is symmetric and non-degenerated.

b) Write down an explicit Gram matrix of $\Lambda^2\widetilde{g}$ in a standard monomial basis of $\Lambda^2\widetilde{g}$ built from an orthonormal[1] basis of V

Set 2. Plücker – Segre – Veronese interaction.

AG2⋄7. In the assumptions and notations of prb. AG2⋄3 and prb. AG2⋄6 take $g(A) = \det A$ as the quadratic form on the space $V = \operatorname{Hom}(U_-, U_+)$. Write $\Lambda^2 g$ for the smooth quadratic form on Λ^2V that sends $v_1 \wedge v_2$ to the Gram determinant $\det \begin{pmatrix} \widetilde{g}(v_1,v_1) & \widetilde{g}(v_1,v_2) \\ \widetilde{g}(v_2,v_1) & \widetilde{g}(v_2,v_2) \end{pmatrix}$ and write $P = \{\omega \in \Lambda^2V \mid \omega \wedge \omega = 0\} \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2V)$ for the Plücker quadric. Show that

a) the intersection of quadrics $V(\Lambda^2 g) \cap P \subset \mathbb{P}_5$ consists of all lines in $\mathbb{P}_3 = \mathbb{P}(V)$ tangent to the Segre quadric $G = V(g) \subset \mathbb{P}_3$.

[1] that is, having the unit Gram matrix

b) the Plücker embedding $\mathrm{Gr}(2,V) \simeq P \subset \mathbb{P}(\Lambda^2 V)$ sends two line rulings of the Segre quadric G to a pair of distinct smooth conics $C_\pm \subset P$ that are cut out of the Plücker quadric by a pair of complementary planes $\Lambda_- = \mathbb{P}\left(S^2U_-^* \otimes \Lambda^2 U_+\right)$ and $\Lambda_+ = \mathbb{P}\left(\Lambda^2 U_-^* \otimes S^2 U_+\right)$ embedded into $\mathbb{P}(\Lambda^2 \mathrm{Hom}(U_-, U_+))$ via prb. AG2◇3

c) both conics $C_- \subset \mathbb{P}\left(S^2U_-^* \otimes \Lambda^2 U_+\right)$ and $C_+ \subset \mathbb{P}\left(\Lambda^2 U_-^* \otimes S^2 U_+\right)$ are the images of the Veronese embeddings $\mathbb{P}(U_-^*) \subset \mathbb{P}\left(S^2 U_-^*\right)$ and $\mathbb{P}(U_+) \subset \mathbb{P}\left(S^2 U_+\right)$, i.e. we have the following commutative diagram of the Plücker – Segre – Veronese interactions[2]:

$$\begin{tikzcd}
\mathbb{P}(U_+) \arrow[rr, hook, "\text{Veronese}"'] & & \mathbb{P}(S^2U_+) \simeq \Lambda_+ \arrow[d, hook] \\
\mathbb{P}_1^+ \times \mathbb{P}_1^- \arrow[u, "\pi_+"] \arrow[d, "\pi_-"] \arrow[r, "\text{Segre}", "\sim"'] & G \subset \mathbb{P}\mathrm{Hom}(U_-, U_+) \arrow[r, dashed, "\text{Plücker}"] & P \subset \mathbb{P}\begin{pmatrix} \Lambda^2 U_-^* \otimes S^2 U_+ \\ \oplus \\ S^2U_-^* \otimes \Lambda^2 U_+ \end{pmatrix} \\
\mathbb{P}(U_-^*) \arrow[rr, hook, "\text{Veronese}"] & & \mathbb{P}(S^2U_-^*) \simeq \Lambda_- \arrow[u, hook]
\end{tikzcd}$$

d) (Hodge's star) Associated with smooth quadretic form g on V is *the Hodge star-operator*

$$* : \Lambda^2 V \xrightarrow{\omega \mapsto \omega^*} \Lambda^2 V\,,$$

defined by prescription $\forall\, \omega_1, \omega_2 \in \Lambda^2 V \quad \omega_1 \wedge \omega_2^* = \Lambda^2 \widetilde{g}(\omega_1, \omega_2) \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4$, where $e_1, e_2, e_3, e_4 \in V$ is an orthonormal basis for g. Verify that this definition does not depend on a choice of orthonormal basis, find eigenvalues and eigenspaces of $*$, and show their place in the previous picture.

Honorary problems

AG2◇8. Generalise prb. AG2◇1 onto the variety of singular quadrics in $\mathbb{P}_n$ for any n.

AG2◇9. Prove the following Taylor expansion for the polynomial $\det(A)$ on the space of $n \times n$-matrices:

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \lambda^p \mu^q \cdot \mathrm{tr}\left(\Lambda^p A \cdot \Lambda^q B^t\right)\,,$$

where $\Lambda^p A$, $\Lambda^q B$ are the matrices of operators induced by A, B on the spaces of homogeneous grassmannian polynomials of degrees p, q (matrix elements of $\Lambda^p A$, $\Lambda^q B$ are $p \times p$ and $q \times q$ minors of A, B numbered in such a way that complementary minors have equal numbers).

AG2◇10 (De Rahm's and Koszul's complexes). Choose a basis $e_1, e_2, \dots, e_n \in V$ and write $x_i \in SV$, $\xi_i \in \Lambda V$ for the classes of e_i in symmetric and exterior algebras respectively. Let $A = \Lambda V \otimes SV$. Consider two linear mappings: *the De Rahm differential* $d = \sum \xi_\nu \otimes \frac{\partial}{\partial x_\nu} : A \to A$ that takes $\omega \otimes f \mapsto \sum\limits_\nu \xi_\nu \wedge \omega \otimes \frac{\partial f}{\partial x_\nu}$ and *the Koszul differential* $\partial = \sum \frac{\partial}{\partial \xi_\nu} \otimes x_\nu : A \to A$ that takes $\omega \otimes f \mapsto \sum\limits_\nu \frac{\partial \omega}{\partial \xi_\nu} \otimes x_\nu \cdot f$.

a) Show that d and ∂ do not depend on a choice of basis and satisfy $d^2 = 0$, $\partial^2 = 0$.

b) Compute $d\partial + \partial d$.

c) (Poincare lemma) Show that both *homology spaces* $\ker d \,/\, \mathrm{im}\, d$ and $\ker \partial \,/\, \mathrm{im}\, \partial$ are 1-dimensional, exhausted by the classes of constants $\Bbbk \cdot 1 \otimes 1 \subset A$.

[2] Plücker is dashed, because it takes lines to points