Set 1. Convexity.

AG3⋄1 (centre of mass). Show that **a)** for any collection of points $Q_1, Q_2, \dots, Q_m \in \mathbb{A}^2$ and any collection of constants[1] $\mu_1, \mu_2, \dots, \mu_m \in \Bbbk$ such that $\sum\limits_{i=1}^{m} \mu_i = \mu \neq 0$ there exists a unique point $M \in \mathbb{A}^2$ such that $\mu_1 \overrightarrow{MQ}_1 + \mu_2 \overrightarrow{MQ}_2 + \dots + \mu_m \overrightarrow{MQ}_m = 0$. **b)** for any point $P \in \mathbb{A}^2$ the point M equals $M = P + \sum_{i=1}^{m} \frac{\mu_i}{\mu} \cdot \overrightarrow{PQ}_i$.

AG3⋄2 (grouping masses). Let a finite collection of points Q_i with masses $\mu_i \in \Bbbk$ and a finite collection of points T_j with masses ν_j have centres of mass at points M and N respetively. Assume that all three sums $\sum \mu_i$, $\sum \nu_j$, $\sum \mu_i + \sum \nu_j$ are non-zeros. Show that centre of mass for the union of all points[2] Q_i and T_j coincides with the centre of mass of two points M , N , taken with the masses $\sum \mu_i$ and $\sum \nu_j$.

AG3⋄3. Give an example of a closed figure Φ with non-empty interior Φ° such that $\overline{\Phi^\circ} \neq \Phi$. Is it possible, if Φ is convex?

AG3⋄4. Give an example of a closed convex figure with a non-closed set of **a)** vertexes[3] **b)** extremal points[4].

Set 2. Polyhedrons and cones.

AG3⋄5. Let a convex polyhedral cone $\sigma \in \mathbb{R}^3$ span the whole vector space. Show that σ and σ^vee have the same number of 1-dimensional edges. Give an example of polyhedral cone $\sigma \in \mathbb{R}^4$ such that σ and $\sigma^\vee$ have different numbers of 1-dimensional edges.

AG3⋄6. Show that a convex subspace η of a convex polyhedral cone σ is a face iff the following equivalence holds: $\forall\ v_1, v_2 \in \sigma\ v_1 + v_2 \in \eta \iff v_1, v_2 \in \eta$.

AG3⋄7. Show that any proper face τ of a convex polyhedral cone σ : **a)** is contained in some hyper-face[5] of σ **b)** coincides with the intersection of all hyper-faces of σ containing τ.

AG3⋄8. Let a convex polyhedral cone $\sigma \subsetneq V$ be generated by vectors $v_1, v_2, \dots, v_N$ that linearly span V , and $\dim V = n$. Prove that: **a)** the boundary $\partial\sigma$ is a union of all hyper-faces of σ

- **b)** covectors $\xi_\tau \in V^*$ annihilating the hyper-surfaces $\sigma \subset \tau$ are contained in a finite set $M \subset V^*$ described as follows: list all the linearly independent collections of $(n-1)$ vectors v_ν; for each such collection find $\xi \in V^*$ that spans its annihilator; if for all generators v_i , $1 \leqslant i \leqslant N$, $\langle\, \xi\,,\, v_i \,\rangle > 0$, then include ξ in M, else if for all v_i $\langle\, \xi\,,\, v_i \,\rangle < 0$, then include $-\xi$ in M, otherwise omit this ξ.
- **c)** $\sigma = \bigcap\limits_{\tau} H^+_{\xi_\tau}$, where $\tau \subset \sigma$ runs through the hyper-faces of σ.

AG3⋄9. Let $\xi \in \sigma^\vee$ and $\tau = \mathrm{Ann}\,(\xi) \cap \sigma$. Prove that $\tau^\vee = \{\zeta - \lambda\xi \mid \zeta \in \sigma^\vee,\ \lambda \geqslant 0\}$.

AG3⋄10. Let convex polyhedral cones σ_1 and σ_2 intersect each other precisely along a common face τ. Show that there exists $\xi \in \sigma_1^\vee \cap (-\sigma_2)^\vee$ such that $\tau = \sigma_1 \cap \mathrm{Ann}\,(\xi) = \sigma_2 \cap \mathrm{Ann}\,(\xi)$.

AG3⋄11. Show that any two vertexes of any convex polyhedron are connected by some pass formed from 1-dimensional edges.

AG3⋄12. Assume that a convex polyhedron $M \subset \mathbb{A}(V)$ does not contain affine subspaces of positive dimension. For each vertex $p \in M$ write $\sigma_p \subset V$ for a cone spanned by all the edges of M outgoing from p. Show that: **a)** $M_\infty = \bigcap\limits_{p} \sigma_p$ **b)** $M \subset p + \sigma_p$ for ny vertex p .

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[1] these constants are called «masses»

[2] «union» of coinciding points means adding their masses

[3] recall that *a vertex* of a convex figure is a face of dimension zero, that is one point intersection with some supporting hyperplane

[4] recall that a point p of a convex figure Φ is called *extremal* if there are no segments $[a, b] \subset \Phi$ such that p is an interior point of $[a, b]$

[5] that is a face of codimension 1

AG3⋄13. Let $M \subset \mathbb{A}(V)$ be a convex polyhedron with vertexes and covector $\xi \in V^*$ be bounded below on M. Show that: **a)** there exist a vertex $p \in M$ such that $\forall\, x \in M\ \langle\, \xi\,,\, x\,\rangle \geqslant \langle\, xi\,,\, p\,\rangle$
b) a vertex $p \in M$ satisfies the above property iff $\langle\, \xi\,,\, q\,\rangle \geqslant \langle\, xi\,,\, p\,\rangle$ for each edge $[p, q] \subset M$ outgoing from p (including those having q at infinity).

Honorary problems

AG3⋄14 (Caratheodori's lemma). Show that each point of the convex hull of an arbitrary figure $\Phi \subset \mathbb{R}^n$ is a convex combination of at most $(n+1)$ points of Φ.

AG3⋄15 (Rhadon's lemma). Show that any finite set of $\geqslant (n+2)$ distinct points in $\mathbb{R}^n$ is a disjoint union of two non-empty subsets with intersecting convex hulls.

AG3⋄16 (Helly's theorem). Given a finite collection of closed convex figures in $\mathbb{R}^n$ such that at least one of them is compact and any $(n+1)$ figures have non-empty intersection, show that the intersection of all the figures is non-empty.

Regular polyhedrons. Given a polyhedron $M \subset \mathbb{R}^n$, *a group of* M is defined as a group of all bijections $M \xrightarrow{\sim} M$ induced by all euclidean linear automorphisms[6] of $\mathbb{R}^n$. Any sequence: vertex of M, edge of M outgoing from this vertex, 2-dimensional face of M outgoing from this edge, … , a hyper-face of M outgoing from theis $(n-1)$-dimensional face, M itself (all intermediate dimensions have to appear) is called *a flag* of M. A polyhedron M is called *regular*, if the group of M acts transitively on the flags of M. Given a regular polyhedron $P \subset \mathbb{R}^n$, we write $\ell = \ell(P)$ for the length of its edge, write $r = r(P)$ for the radius of its superscribed sphere, and put $\varrho = \varrho(P) \stackrel{\text{def}}{=} \ell^2/4r^2$. In all the problems below assume that a regular polyhedron $P \subset \mathbb{R}^n$ linearly spans the whole vector space.

AG3⋄17 (the star). Show that all vertexes of P joint with a given vertex $p \in P$ by an edge of P form a regular polyhedron in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. It is called *a star* of P and is denoted by $\mathrm{st}(P)$.

AG3⋄18 (the symbol). *Schläfli's symbol* of a regular polyhedron $P \subset \mathbb{R}^n$ is a collection of $(n-1)$ positive integers $\boldsymbol{\nu}(P) = \big(\nu_1(P),\, \nu_2(P),\, \dots\,,\, \nu_{n-1}(P)\big)$, defined inductively as follows: $\nu_1(P)$ equals the number of edges of 2-dimensional face of P and the rest sub-sequence $\big(\nu_2(P),\, \dots\,,\, \nu_{n-1}(P)\big) = \boldsymbol{\nu}\big(\mathrm{st}(P)\big)$ is the Schläfli symbol of the star $\mathrm{st}(P)$. Find the symbols of regular: **a)** dodecahedron in $\mathbb{R}^3$ **b)** icosahedron in $\mathbb{R}^3$ **c)** n-dimensional simplex **d)** n-dimensional cube **e)** n-dimensional cocube[7].

AG3⋄19. Express $\ell(\mathrm{st}(P))$ through $\ell(P)$ and $\nu_1(P)$.

AG3⋄20. Show that $\varrho(P)$ depends only on the symbol of P and satisfies the equality

$$\varrho(P) = 1 - \big(\cos^2\big(\pi/\nu_1(P)\big)\big) / \big(\varrho\big(\mathrm{st}(P)\big)\big)\ .$$

AG3⋄21 (duality). Let $P \subset \mathbb{R}^n$ be a regular polyhedron with the centre at the origin. **a)** Show that $P^* = \{\xi \in \mathbb{R}^{n*} \mid \xi(v) \geqslant -1\ \ \forall\, v \in P\ \}$ a regular polyhedron with the centre at the origin. **b)** For each k construct a canonical bijection between k-dimensional spaces of P and $(n-k-1)$-dimensional faces of P^* reversing the inclusions of faces. **c)** Prove that the symbol of P^* is the symbol of P read from the right to the left.

AG3⋄22 (clasification of regular polyhedrons). Show that the symbols of all regular polyhedrons $P \subset \mathbb{R}^n$ are contained in the following list:
a) (ν), where $\nu \geqslant 3$ is any positive integer, for $n = 2$
b) $(3,\ 3)$, $(3,\ 4)$, $(4,\ 3)$, $(3,\ 5)$, $(5,\ 3)$, for $n = 3$
c) $(3,\ 3,\ 3)$, $(3,\ 3,\ 4)$, $(4,\ 3,\ 3)$, $(3,\ 4,\ 3)$, $(3,\ 3,\ 5)$, $(5,\ 3,\ 3)$, for $n = 4$
d) $(3,\ \dots\,,\ 3)$, $(3,\ \dots\,,\ 3,\ 4)$, $(4,\ 3,\ \dots\,,\ 3)$ for $n \geqslant 5$
and for each symbol in the list there exists a unique up to dilatation regular polyhedron that has this symbol.

[6]we asume that $\mathbb{R}^n$ is equipped with the standard euclidean structure $|x|^2 = \sum x_i^2$
[7]that is, the convex hull of centres of the hyper-faces of the cube