Program of an intensive 3-weeks course in Linear Geometry and supplying Linear Algebra for São – Carlos branch of the Universidade de São Paulo

1st WEEK — PROJECTIVE WEEK

1st DAY: NUMBERS, FUNCTIONS, POINTS, AND VARIETIES

- Numbers & Vectors. Fields (especially finite ones). Vector spaces and bases: samples of usage. Could a field of 27 elements contain a subfield of 9 elements?
- Points and figures. Affine space. Polynomial functions. Algebraic varieties. Given $\dim V$, find $\dim S^n V$.
- Projective space. Cell decomposition and affine charts. Topological degression: $\mathbb{RP}_1 = S^1$, $\mathbb{CP}_1 = S^2$, $\mathbb{RP}_2 = D^2 \underset{S^1}{\#} \texttt{Möbius tape}$, $\pi_1(\mathrm{SO}_3 = \mathbb{RP}_3) = \mathbb{Z}/(2)$. Projective varieties.
- Space of hypersyrfaces. Basic example: $S^d\mathbb{P}_1 = \mathbb{P}_d$. Veronese curve. How to present Veronese varieties by quadratic equations
- Projections. Complementary subspaces and joins. Projecting Quadric, rational parametrization of conic. Projecting Veronese curve: rational curves, smooth cubic is not rational.

2D DAY: QUADRICS

- Projective linear group, especially PGL_2. Coss-ratio, quadrangle, the epimorphism $\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$, harmonic pairs of points. Homographies: games with lines, games with conics, Pascal's theorem, Ponselet's porism. Space of conics, pencils of conics.
- Projective quadrics: polarization, correlation, tangent space, smoothness, polar mapping. Linear subspaces on a smooth quadric. Space of quadrics. Segre's quadric in $\mathbb{P}_3$, Schläflische doppelsechs. How many lines do cross 4 given mutually skew lines in 3D-space?
- Orthogonal geometry of quadratic form over an arbitrary field. Orthogonal group is spanned by reflections. Orthogonal decomposition into hyperbolic and anisotropic parts.

2d WEEK — TENSOR WEEK

1st day: non-commutative polynomials

- Tensor products and Segre varieties. Tensor algebra of a vector space: contractions, linear span of a tensor.
- Grassmannian algebra. Linear change of basis and Laplace relations. Grassmannian quadratic forms. Polarization and partial derivatives. Linear span of skew polynomial and Plücker relations.
- $\mathrm{Gr}(2,4)$ and lines in $\mathbb{P}_3$. Plücker quadric in $\mathbb{P}_5$. Planes and lines on the Plücker quadric. Schubert cells.

2d day: grassmanians

- Plücker – Segre – Veronese interaction and spinors for SO_4 and SO_6.
- Spinors for SO_5 and the lagrangian grassmannian $\mathrm{LGr}(2,4)$
- Grassmannians in general: cell decomposition, affine charts, Plücker embedding, introduction to the Schubert calculus.

3d WEEK: AFFINE WEEK

1st day: convexity and polyhedra

- Barycentric combinations, barycentric coordinates as local affine coordinates on projective space. Convexity. Interaction with topology
- Supporting hyperplanes and properties of affine-linear maps.
- Convex figures: faces and extreme points. Convex hulls vs intersections of half-spaces.
- Compact convex polytopes, Minkowski – Weyl theorem.

2d day: polyhedra vs cones

- Convex polyhedral cones, Farkas lemma and variations.
- Gale duality and variations.
- Geometric aspects of linear optimization.