Final Written Exam

The problems may be solved in any order. The complete solution of each problem is worth 10 points. Only an answer without its explanation gives 0 points irrespective of its correctness. To get the 100% result it is enough to collect 50 points in sum, e.g., solve completely any 5 of the 7 problems.

By default, the ground field is assumed to be algebraically closed of zero characteristic.

Problem 1 (10 points). Let $C_3 \subset \mathbb{P}_3$ be a rational normal cubic[1]. Is it true or not that for every point $p \in \mathbb{P}_3 \smallsetminus C_3$, there exists a line $\ell \subset \mathbb{P}_3$ passing through p and intersecting C_3 in at least two (possibly coinciding) points? If such a line exists for some p, should it be unique?

Problem 2 (10 points). Let C_1, C_3, C_3 be the three split conics in a simple pencil L of conics on $\mathbb{P}_2$ and a, b, c, d the four base points of L. For an arbitrary conic $C \in L$, compare the cross-ratios $[C_1, C_2, C_3, C]$ on L and $[a, b, c, d]$ on C.

Problem 3 (10 points). Given two quadrics $G, Q \subset \mathbb{P}_n$, not necessary smooth, show that the pencil of quadrics spanned by them contains exactly $(n+1)$ distinct singular quadrics if and only if G and Q are transversal, that is, $\dim(T_pG \cap T_pQ) = n-2$ for all $p \in G \cap Q$.

Problem 4 (10 points). Write M for the projective space of nonzero $m \times n$ matrices considered up to proportionality. Use appropriate incidence variety $\Gamma = \{(L, F) \mid L \subset \ker F\}$, where L is a subspace and F is a matrix, to show that matrices of rank at most k form an irreducible projective subvariety $M_k \subset M$, and find $\dim M_k$.

Problem 5 (10 points). Show that the lines laying on a smoth quadric in $\mathbb{P}_4$ form a closed irreducible subvariety in the Grassmannian of all lines in $\mathbb{P}_4$, and find the dimension of this subvariety.

Problem 6 (10 points). Given two projective algebraic varieties $X, Y \subset \mathbb{P}(V)$, write $\mathcal{J}(X, Y) \subset \mathrm{Gr}(2, V)$ for the Zariski closure of the set of lines[2] (xy) joining distinct points $x \in X$, $y \in Y$, and $J(X, Y) \subset \mathbb{P}(V)$ for the union of lines $\ell \subset \mathbb{P}(V)$, $\ell \in \mathcal{J}(X, Y)$. Show that $J(X, Y)$ is Zariski closed. May $J(X, Y)$ be reducible for irreducible X, Y? Find $\dim J(X, Y)$ for irreducible non-intersecting X, Y of given dimensions.

Problem 7 (10 points). Given six points $p_1, p_2, \dots, p_6 \in \mathbb{P}_2 = \mathbb{P}(V)$ any three of which are noncollinear and all the six do not lie on a common conic, let $W \subset \mathbb{P}(S^3V^*)$ be the projective space of all cubic curves on $\mathbb{P}_2$ passing through the given points. Consider the map $\mathbb{P}_2 \smallsetminus \{p_1, p_2, \dots, p_6\} \to W^\times$ that sends a point p to the hyperplane in W formed by all cubics passing through p. Write $S \subset W^\times$ for the closure of the image of this map. Show that $S \subset W^\times$ is a smooth cubic surface in $\mathbb{P}_3$ and describe the 27 pencils of cubic curves passing through $\{p_1, p_2, \dots, p_6\}$ dual to the 27 lines laying on S.

[1] That is, a curve congruent to the Veronese cubic modulo linear projective automorphisms of $\mathbb{P}_3$.

[2] Considered as the points of the Grassmannian $\mathrm{Gr}(2, V)$ of all lines in $\mathbb{P}(V)$.