§2 Projective Quadrics

2.1 Quadratic forms and quadrics. We assume on default in §2 that char $k \neq 2$. Projective hypersurfaces of degree 2 are called projective quadrics. Given a non-zero quadratic form $q \in S^2V^*$, we write $Q \subset \mathbb{P}(V)$ for the quadric $Q = V(q)$.

2.1.1 The Gram matrix. If char $k \neq 2$, then every quadratic form $q \in S^2V^*$ on $V = \mathbb{K}^{n+1}$ can be written as $q(x) = \sum_{i,j} a_{ij} x_i x_j = x A x^t$, where $x = (x_0, x_1, \ldots, x_n)$ is the coordinate row, $x^t$ is the transposed column of coordinates, and $A = (a_{ij}) \in \text{Mat}_{n+1}(k)$ is a symmetric square matrix. Every non-diagonal element $a_{ij} = a_{ji}$ of $A$ equals the half $^3$ of coefficient of monomial $x_i x_j$ in the reduced expansion for $q$. The matrix $A$ is called the Gram matrix of $q$ in the chosen basis of $V$.

In other words, for any quadratic polynomial $q$ on $V$, there exists a unique symmetric bilinear form $\tilde{q} : V \times V \to k$ such that $q(v) = \tilde{q}(v, v)$ for all $v \in V$. In coordinates, $\tilde{q}(x, y) = \sum_{i,j} a_{ij} x_i x_j = x A y^t = \frac{1}{2} \sum_{i} y_i \frac{\partial q(x)}{\partial x_i}$.

Exercice 2.1. Check this.

The symmetric bilinear form $\tilde{q}$ is called the polarization of quadratic form $q$. It can be thought of as an inner product on $V$, possibly degenerated. The elements of Gram matrix equal the inner products of basic vectors: $a_{ij} = \tilde{q}(e_i, e_j)$. In the matrix notations, $A = e^t \cdot e$, where $e = (e_0, e_1, \ldots, e_n)$ is the row of basic vectors in $V$, $e^t$ is the transposed column of basic vectors, and $u \cdot w \equiv \tilde{q}(u, w) \in k$ for $u, w \in V$. When we pass to another basis $e' = eC$, where $C \in \text{GL}_{n+1}(k)$, the Gram matrix $A$ of $e$ is related with the Gram matrix $A'$ of $e'$ as $A' = C^t AC$, because $(e')^t \cdot e' = C^t e^t \cdot eC$.

2.1.2 The Gram determinant. Since $\det A' = \det A \cdot \det^2 C$, the determinant of Gram matrix does not depend on the choice of basis up to multiplication by non zero squares from $k$. We write $\det q \equiv \det A \cdot \det^2 C$ for the class of $\det A$ modulo multiplication by non zero squares, and call it the Gram determinant of quadratic form $q \in S^2V^*$. The form $q$ and quadric $Q = V(q)$ are called smooth or non-degenerated, if $\det q \neq 0$. Otherwise they are called singular or degenerated.

2.1.3 The rank. Since the rank of matrix is not changed under multiplications of the matrix by non-degenerated matrices, the rank of Gram matrix does not depend on the choice of basis as well. It is called the rank of quadratic form $q$ and quadric $Q = V(q)$, and denoted by $rk q = rk Q \equiv rk A$.

Proposition 2.1 (LAGRANGE’S THEOREM)
For any quadratic form $q$ there exists a basis where the Gram matrix of $q$ is diagonal.

Proof. Induction on dim$V$. If $q \equiv 0$ or dim$V = 1$, then the Gram matrix is diagonal. If dim$V \geq 2$ and $q(e) = \tilde{q}(e, e) \neq 0$ for some $e \in V$, we put $e_1 = e$ to be the first vector of desired basis. Every vector $v \in V$ admits a unique decomposition $v = \lambda e + u$, where $\lambda \in k$ and $u \in V^\perp = \{ w \in V | \tilde{q}(v, w) = 0 \}$. Indeed, the orthogonality of $v$ and $v - \lambda e$ forces $\lambda = \tilde{q}(e, v) / \tilde{q}(e, e)$, then $u = v - (\tilde{q}(e, v) / \tilde{q}(e, e)) \cdot e$.

Exercise 2.2. Verify that $v - (\tilde{q}(e, v) / \tilde{q}(e, e)) \cdot e \in V^\perp$.
Thus, we have the orthogonal decomposition $V = \mathbb{K} \cdot e \oplus e^\perp$. By induction, there exists a basis $e_2, \ldots, e_n$ in $e^\perp$ with diagonal Gram matrix. Hence, $e_1, e_2, \ldots, e_n$ is a required basis for $V$.

\footnote{Note that if char $k = 2$, such the matrix $A$ does not always exists.}
COROLLARY 2.1
Every quadratic form $q$ over an algebraically closed field turns to the sum of squares
$$q(x) = x_0^2 + x_1^2 + \cdots + x_k^2, \text{ where } k + 1 = \text{rk } q,$$
in appropriate coordinates on $V$.

PROOF. Pass to a basis $e_0, e_1, \ldots, e_n$ in which the Gram matrix is diagonal, renumber the vectors $e_i$ in order to have $q(e_i) \neq 0$ exactly for $1 \leq i \leq k$, then multiply all these $e_i$ by $1/\sqrt{q(e_i)} \in k$. □

EXAMPLE 2.1 (QUADRICS ON $\mathbb{P}_1$)
It follows from the Proposition 2.1 that the equation of any quadric $Q \subset \mathbb{P}_1$ can be written in appropriate coordinates on $\mathbb{P}_1$ either as $x_0^2 = 0$ or as $x_0^2 + ax_1^2 = 0$, where $a \neq 0$. In the first case, $Q$ is singular, $\text{rk } Q = 1$, and the equation of $Q$ is the squared linear equation of the point $(0 : 1)$. By this reason, such a quadric is called a **double point**. In the second case, $\text{rk } Q = 2$, the quadric is smooth, and its Gram determinant equals $a$ up to multiplication by non-zero squares.

If $-\alpha \in k$ is not a square, then the equation $(x_0 / x_1)^2 = -\alpha$ has no solutions, and the quadric is empty. If $-\alpha = \delta^2$ for some $\delta \in k$, then $x_0^2 + ax_1^2 = (x_0 - \delta x_1)(x_0 + \delta x_1)$ has two distinct roots $(\pm \delta : 1) \in \mathbb{P}_1$. Thus, the geometry of quadric $Q = V(q) \subset \mathbb{P}_1$ is completely determined by the Gram determinant $q \in k^\times$. If $q = 0$, then the quadric is a double point. If $-\delta q = 1$, that is, $-\delta A \in k$ is a square, then the quadric consists of two distinct points. If $-\delta q \neq 1$, that is, $-\delta A \notin k$ is not a square, then the quadric is empty. Note that the latter case never appears over an algebraically closed field $k$.

2.2 **Tangent lines.** It follows from the Example 2.1 that there are precisely 4 different positional relationships between a quadric $Q$ and a line $\ell$ in $\mathbb{P}_n$: either $\ell \subset Q$, or $\ell \cap Q$ is a double point, or $\ell \cap Q$ is a pair of distinct points, or $\ell \cap Q = \emptyset$, and the latter case never appears over an algebraically closed field.

DEFINITION 2.1 (TANGENT SPACE OF QUADRIC)
A line $\ell$ is called **tangent** to a quadric $Q$ at a point $p \in Q$, if either $p \in \ell \subset Q$ or $Q \cap \ell$ is the double point $p$. In these cases we say that $\ell$ **touches** $Q$ at $p$. The union of all tangent lines touching $Q$ at a given point $p \in Q$ is called the **tangent space** to $Q$ at $p$ and denoted by $T_p Q$.

PROPOSITION 2.2
A line $(ab)$ touches a quadric $Q = V(q)$ at the point $a \in Q$ if and only if $\tilde{q}(a, b) = 0$.

PROOF. The Gram matrix of restriction $q|_{(a, b)}$ in the basis $a, b$ of line $(ab)$ is
$$
\begin{pmatrix}
\tilde{q}(a, a) & \tilde{q}(a, b) \\
\tilde{q}(a, b) & \tilde{q}(b, b)
\end{pmatrix}.
$$
Since $\tilde{q}(a, a) = q(a) = 0$ by assumption, the Gram determinant $\det q|_{(a, b)} = \tilde{q}(a, b)^2$. It vanishes if and only if $\tilde{q}(a, b) = 0$. □

COROLLARY 2.2 (APPARENT CONTOUR OF QUADRIC)
For any point $p \notin Q$, the **apparent contour** of $Q$ viewed from $p$, i.e., the set of all points $a \in Q$ such that the line $(pa)$ touches $Q$ at $a$, is cut out $Q$ by the hyperplane $\Pi_p \equiv \{ x \in \mathbb{P}_n | \tilde{q}(p, x) = 0 \}$.

PROOF. Since $\tilde{q}(p, p) = q(p) \neq 0$, the equation $\tilde{q}(p, x) = 0$ is a non-trivial linear homogeneous equation on $x$. Thus, $\Pi_p \subset \mathbb{P}_n$ is a hyperplane, and $Q \cap \Pi$ coincides with the apparent contour of $Q$ viewed from $p$ by the Proposition 2.2. □
2.2.1 Smooth and singular points. Associated with a quadratic form \( q \in S^2 V^* \) is the linear mapping

\[
\hat{q} : V \to V^*, \quad v \mapsto \hat{q}(*, v).
\]

sending a vector \( v \in V \) to the linear form \( \hat{q}(v) : V \to \mathbb{K}, \ w \mapsto \hat{q}(w, v) \). The map (2-2) is called the \textit{correlation} of quadratic form \( q \).

Exercise 2.3. Convince yourself that the matrix of linear map (2-2) written in dual bases \( e, x \) of \( V \) and \( V^* \) coincides with the Gram matrix of \( q \) in the basis \( e \).

This shows once more, that the rank \( \text{rk} A = \dim V - \dim \ker \hat{q} \) does not depend on a choice of basis.

The vector space \( \ker(q) \triangleq \ker \hat{q} = \{ v \in V \mid \hat{q}(w, v) = 0 \ \forall w \in V \} \) is called \textit{the kernel} of quadratic form \( q \). The projectivization of the kernel is denoted

\[
\text{Sing} \ Q \triangleq \mathbb{P}(\ker q) = \{ p \in \mathbb{P}(V) \mid \forall u \in V \ \hat{q}(p, u) = 0 \}
\]

and called \textit{the vertex space} or \textit{the singular locus} of quadric \( Q = V(q) \subset \mathbb{P}_n \). The points of \( \text{Sing} \ Q \) are called \textit{singular}. All points of the complement \( Q \setminus \text{Sing} \ Q \) are called \textit{smooth}. Thus, a point \( p \in Q \subset \mathbb{P}(V) \) is smooth if and only if the tangent space \( T_p Q = \{ x \in \mathbb{P}_n \mid \hat{q}(p, x) = 0 \} \) is a hyperplane in \( \mathbb{P}_n \).

Conversely, a point \( p \in Q \subset \mathbb{P}(V) \) is singular if and only if the tangent space \( T_p Q = \mathbb{P}(V) \) is the whole space, that is, any line passing through \( \alpha \) either lies on \( Q \) or does not intersect \( Q \) anywhere besides \( \alpha \).

Exercise 2.4. Convince yourself that the singularity of a point \( p \in Q \subset \mathbb{P}_n \) means that

\[
\frac{\partial q}{\partial x_i}(p) = 0 \quad \text{for all} \ 0 \leq i \leq n.
\]

Note that a quadric is smooth in the sense of \( n^* \text{2.1.2} \) if and only if it has no singular points.

Lemma 2.1
If a quadric \( Q \subset \mathbb{P}_n \) has a smooth point \( \alpha \in Q \), then \( Q \) is not contained in a hyperplane.

Proof. For \( n = 1 \), this follows from \textit{the Example 2.1}. Consider \( n \geq 2 \). If \( Q \) lies inside a hyperplane \( H \), then every line \( \ell \nsubseteq H \) passing through \( \alpha \) intersects \( Q \) only in \( \alpha \) and therefore is tangent to \( Q \) at \( \alpha \). Hence, \( \mathbb{P}_n = H \cup T_p Q \). This contradicts to \textit{the Exercise 2.5} below.

Exercise 2.5. Show that the projective space over a field of characteristic \( \neq 2 \) is not a union of two hyperplanes.

Theorem 2.1
For any quadric \( Q \subset \mathbb{P}(V) \) and projective subspace \( L \subset \mathbb{P}(V) \) complementary to \( \text{Sing} \ Q \), the intersection \( Q' = L \cap Q \) is a smooth quadric in \( L \), and \( Q \) is the linear join\(^1\) of \( Q' \) and \( \text{Sing} \ Q \).

Proof. Let \( L = \mathbb{P}(U) \). Then \( V = \ker q \oplus U \). Assume that there exists a vector \( u \in U \) such that \( \hat{q}(u, u') = 0 \) for all \( u' \in U \). Since \( \hat{q}(u, w) = 0 \) for all \( w \in \ker q \) as well, the equality \( \hat{q}(u, v) = 0 \) holds for all \( v \in V \). Hence, \( u \in \ker q \cap U = 0 \). That is, \( Q' \) is smooth. Every line \( \ell \) that intersects \( \text{Sing} \ Q \) but is not contained in \( \text{Sing} \ Q \) does intersect \( L \) and either is contained in \( Q \) or does not intersect \( Q \) anywhere besides the point \( \ell \cap \text{Sing} \ Q \). This forces \( Q \) to be the union of lines \( \{sp\} \) such that \( s \in \text{Sing} \ Q, \ p \in L \cap Q \).

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\(^1\)For sets \( X, Y \subset \mathbb{P}_n \), their linear join is the union of all lines \( (xy) \) such that \( x \in X, y \in Y \).
2.3 Duality. Projective spaces $\mathbb{P}_n = \mathbb{P}(V)$, $\mathbb{P}_n^\times \equiv \mathbb{P}(V^*)$, obtained from dual vector spaces $V, V^*$, are called dual. Geometrically, $\mathbb{P}_n^\times$ is the space of hyperplanes in $\mathbb{P}_n$, and vice versa. The linear equation $\langle \xi, v \rangle = 0$, being considered as an equation on $v \in V$ for a fixed $\xi \in V^*$, defines a hyperplane $\mathbb{P}(\text{Ann } \xi) \subset \mathbb{P}_n$. As an equation on $\xi$ for a fixed $v$, it defines a hyperplane in $\mathbb{P}_n^\times$ formed by all hyperplanes in $\mathbb{P}_n$ passing through $v$. For every $k = 0, 1, \ldots, n$ there is the canonical involutive\(^1\) bijection $L \leftrightarrow \text{Ann } L$ between projective subspaces of dimension $k$ in $\mathbb{P}_n$ and projective subspaces of dimension $(n-k-1)$ in $\mathbb{P}_n^\times$. It is called the projective duality. For a given $L = \mathbb{P}(U) \subset \mathbb{P}_n$, the dual subspace $\text{Ann } L \equiv \mathbb{P}(\text{Ann } U) \subset \mathbb{P}_n^\times$ consists of all hyperplanes in $\mathbb{P}_n$ containing $L$. The projective duality reverses inclusions: $L \subset H \iff \text{Ann } L \supset \text{Ann } H$, and sends intersections to linear joins, and vice versa. This allows to translate the theorems true for $\mathbb{P}_n$ to the dual statements about the dual figures in $\mathbb{P}_n^\times$. The latter may look quite dissimilar to the original. For example, the collinearity of 3 points in $\mathbb{P}_n$ is translated as the existence of codimension-2 subspace common for 3 hyperplanes in $\mathbb{P}_n^\times$.

2.3.1 The polar mapping. For a smooth quadric $Q = V(q)$, the correlation $\hat{q}: V \to V^*$ is an isomorphism. The induced linear projective isomorphism $\overline{q}: \mathbb{P}(V) \to \mathbb{P}(V^*)$ is called the polar mapping or the polarity provided by quadric $Q$. The polarity sends a point $p \in \mathbb{P}_n$ to the hyperplane

$$
\Pi_p = \text{Ann } \overline{q}(p) = \{ x \in \mathbb{P}(V) \mid \overline{q}(p, x) = 0 \},
$$

which cuts apparent contour of $Q$ viewed from $p$ in accordance with the Corollary 2.2. The hyperplane $\Pi_p$ and point $p$ are called the polar and pole of one other with respect to $Q$. If $p \in Q$, then $\Pi_p = T_p Q$ is the tangent plane to $Q$ at $p$. Note that $a$ lies on the polar of $b$ if and only if $b$ lies on the polar of $a$, because the condition $\overline{q}(a, b) = 0$ is symmetric. Such points $a, b$ are called conjugated with respect to the quadric $Q = V(q)$.

**Proposition 2.3**

Let a line $(ab)$ intersect a smooth quadric $Q$ in two distinct points $c, d$ different from $a, b$. Then $a, b$ are conjugated with respect to $Q$ if and only if they are harmonic to $c, d$.

**Proof.** Chose some homogeneous coordinate $x = (x_0 : x_1)$ on the line $\ell = (ab) = (cd)$. The intersection $Q \cap \ell = \{c, d\}$ considered as a quadric in $\ell$ is the zero set of quadratic form

$$
q(x) = \det(x, c) \cdot \det(x, d),
$$

whose polarization is $\overline{q}(x, y) = \frac{1}{2} \left( \det(x, c) \cdot \det(y, d) + \det(y, c) \cdot \det(x, d) \right)$. Thus, $\overline{q}(a, b) = 0$ means that $\det(a, c) \cdot \det(b, d) = -\det(b, c) \cdot \det(a, d)$, i.e., $[a, b, c, d] = -1$. \qed

**Proposition 2.4**

Let $G, Q \subset \mathbb{P}_n$ be two quadrics with Gram matrices $A, \Gamma$ in some basis of $\mathbb{P}_n$. If $G$ is smooth, then the polar mapping of $G$ sends $Q$ to the quadric $Q_G^\times \subset \mathbb{P}_n^\times$ which has the Gram matrix $A_G^\times = \Gamma^{-1} A \Gamma^{-1}$ in the dual basis of $\mathbb{P}_n^\times$. Note that $\text{rk } Q_G^\times = \text{rk } Q$.

**Proof.** Write the homogeneous coordinates in $\mathbb{P}_n$ as row vectors $x$ and dual coordinates in $\mathbb{P}_n^\times$ as column vectors $\xi$. The polarity $\mathbb{P}_n \to \mathbb{P}_n^\times$ provided by $G$ sends $x \in \mathbb{P}_n$ to $\xi = \Gamma x \xi$. Since $\Gamma$ is invertible, $x$ is recovered from $\xi$ as $x = \xi^\top \Gamma^{-1}$. When $x$ runs through the quadric $x A x \xi = 0$, the corresponding $\xi$ fills the quadric $\xi^\top \Gamma^{-1} A \Gamma^{-1} \xi = 0$. \qed

\(^1\)That is, inverse to itself: $\text{Ann } \text{Ann } L = L$. 
COROLLARY 2.3
The tangent spaces to a smooth quadric $Q \subset \mathbb{P}_n$ form the smooth quadric $Q^\times \subset \mathbb{P}_n^\times$. The Gram matrices of $Q$, $Q^\times$ in dual bases of $\mathbb{P}_n$, $\mathbb{P}_n^\times$ are inverse to each other.

PROOF. Put $G = Q$ and $\Gamma = A$ in the Proposition 2.4. \hfill \square

2.3.2 Polarisations over non-closed fields. If $k$ is not algebraically closed, then there are non-singular quadratic forms $q \in S^2V^2$ with $V(q) = \emptyset$. However, their polars $\overline{q} : \mathbb{P}(V) \to \mathbb{P}(V^*)$, that is, the bijective correspondences between points and hyperplanes, are non-trivial anyway.

EXERCISE 2.6. Describe geometrically the polarity with respect to «imaginary circle» $x^2 + y^2 = -1$ in the Euclidean plane $\mathbb{R}^2$.

Thus, the polarities are much more informative than the quadrics. The quadric is recovered from its polarity as the set of all points lying on the own polars, i.e., the self-conjugated points. It follows from the Theorem 1.1 that two polarities coincide if and only if the corresponding quadratic forms are proportional. Thus, the polars on $\mathbb{P}_n = \mathbb{P}(V)$ stay in bijection with the points of projective space $\mathbb{P}(S^2V^*) = \mathbb{P}_{n+2}/\mathbb{P}_2$. Somewhat erroneous, the latter is called the space of quadrics in $\mathbb{P}(V)$. The quadrics $Q \subset \mathbb{P}_n$ passing through a given point $p \in \mathbb{P}_n$ form a hyperplane in the space of quadrics, because the equation $q(p) = 0$ is linear homogeneous in $q \in \mathbb{P}(S^2V^*)$.

PROPOSITION 2.5
Every collection of $n(n+3)/2$ points in $\mathbb{P}_n$ lies on some quadric.

PROOF. Any $n(n+3)/2$ hyperplanes in $\mathbb{P}_{n+2}/\mathbb{P}_2$ have a non empty intersection. \hfill \square

PROPOSITION 2.6
Over an infinite field, two nonempty smooth quadrics coincide if and only if their equations are proportional.

PROOF. If $V(q_1) = V(q_2)$ in $\mathbb{P}(V)$, then two polarities $\overline{q}_1, \overline{q}_2 : \mathbb{P}(V) \to \mathbb{P}(V^*)$ coincide in all points of the quadrics. It follows from the Corollary 1.1 on p. 12 and the Exercise 2.7 below that the correlation maps $\overline{q}_1, \overline{q}_2 : V \to V^*$ and therefore the Gram matrices are proportional. \hfill \square

EXERCISE 2.7. Check that over an infinite field, every nonempty smooth quadric $\mathbb{P}_n$ contains $n + 2$ points such that no $n + 1$ of them lie within a hyperplane.

2.4 Conics. Plane quadrics are called conics. For $\mathbb{P}_2 = \mathbb{P}(V)$, the space of conics $\mathbb{P}(S^2V^*) = \mathbb{P}_5$. Conics of rank 1 are called a double lines. In appropriate coordinates, such a conic has the equation $x_0^2 = 0$. It is totally singular, i.e., has no smooth points at all. By the Theorem 2.1 on p. 18, a conic $S$ of rank 2 is the linear join of the singular point $s \in S$ and a smooth quadric $S \cap \ell$ within a line $\ell \neq s$. By the Example 2.1 on p. 17, $S \cap \ell$ either consists of two distinct points or is empty. In the first case, $S$ is the union of two lines intersecting at the singular point $s$. Such a conic is called split. If $S \cap \ell = \emptyset$, then $S = \{s\}$ consists of the singular point only. For example, the conic $x_0^2 + x_1^2 = 0$ in $\mathbb{P}(\mathbb{R}^2)$ is of this sort. Over an algebraically closed field, there are no such conics, certainly.

Lemme 2.2 (Rational Parametrization of Non-empty Smooth Conic)
Every non-empty smooth conic $C \subset \mathbb{P}_2$ over any field $k$ with $\text{char } k \neq 2$ admits a rational quadratic parametrization, i.e., there exist homogeneous quadratic polynomials $\varphi_0, \varphi_1, \varphi_2 \in k[t_0, t_1]$ such that the map $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$, $(t_0 : t_1) \mapsto (\varphi_0(t_0, t_1) : \varphi_1(t_0, t_1) : \varphi_2(t_0, t_1))$, establishes a bijection between $\mathbb{P}_1$ and $C$. 
Proof. Given a point \( p \in C \), a required parametrization is provided by the projection \( \varphi : \ell \to C \) of an arbitrary line \( \ell \not\ni p \) from \( p \) onto \( C \). For every \( t \in \ell \), the line \( (pt) \) intersects \( C \) at \( p \) and one more point, which coincides with \( p \), if \( (pt) = T_p C \), and differs from \( p \) for all other \( t \). In the first case we put \( \varphi(t) = a \). For all other \( t \), the second intersection point can be written as \( t + \lambda p \), where \( \lambda \in \mathbb{k} \), and satisfies the equation \( \tilde{q}(t + \lambda p, t + \lambda p) = 0 \), which is equivalent to \( q(t) = -2\lambda \tilde{q}(t, p) \). Thus, the map \( \varphi : \ell \to C \) takes \( t \in \ell \) to \( \varphi(t) = q(t) \cdot p - 2q(p, t) \cdot t \in C \). \( \square \)

Exercise 2.8. Verify that the right hand side of the latter formula equals \( p \) for \( t = T_p C \cap \ell \), and make sure that \( \varphi \) is described in coordinates by a triple of quadratic homogeneous polynomials in the coordinates of \( t \) as required.

Lemma 2.3
The intersection \( C \cap D \) of a smooth conic \( C \) with a curve \( D \) of degree \( d \) in \( \mathbb{P}_2 \) either consists of at most \( 2d \) points or coincides with \( C \).

Proof. Let \( \varphi : \mathbb{P}_1 \to \mathbb{P}_2 \), \( (t_0 : t_1) \mapsto (\varphi_0(t_0, t_1) : \varphi_1(t_0, t_1) : \varphi_2(t_0, t_1)) \) be a rational quadratic parameterization of \( C \), and \( D = V(f) \) for some homogeneous polynomial \( f(x_0, x_1, x_2) \) of degree \( d \). The values of parameter \( t \) corresponding to the intersection point \( C \cap D \) satisfy the equation \( f(\varphi_0(t), \varphi_1(t), \varphi_2(t)) = 0 \), whose left hand side is either the zero polynomial or a non-zero homogeneous polynomial of degree \( 2d \). In the first case \( C \subset D \). In the second case the equation has at most \( 2d \) solutions in \( \mathbb{P}_1 \). \( \square \)

Proposition 2.7
Any 5 points in \( \mathbb{P}_2 \) lie on a conic. Such a conic \( C \) is unique if and only if every 4 of the points are non-collinear. If every 3 of the points are non-collinear, the conic \( C \) is smooth.

Proof. The first statement is exactly the Proposition 2.5 for \( n = 2 \). Let a line \( \ell \) pass through some 3 of the given points. Then any conic \( C \) passing through the given points contains \( \ell \). If the remaining two points \( a, b \) do not lie on \( \ell \), then \( C = \ell \cup (ab) \) is unique. If \( a \in \ell \), then for any line \( \ell' \ni b \), the split conic \( \ell \cup \ell' \) contains all five given points. If any 3 of the given points are non-collinear, then every conic passing through the 5 given points is smooth, because a singular conic is either a line, or a pair of lines, or a point. Since two different smooth conics have at most 4 intersection points by the Lemma 2.3, a smooth conic passing through 5 points is unique. \( \square \)

Corollary 2.4
Any 5 lines without triple intersections in \( \mathbb{P}_2 \) do touch a unique smooth conic.

Proof. This is projectively dual to the last statement in the Proposition 2.7. \( \square \)

2.5 Quadratic surfaces. The space of quadrics in \( \mathbb{P}_3 = \mathbb{P}(V) \) is \( \mathbb{P}(S^2 V^*) = \mathbb{P}_9 \). In particular, any 9 points in \( \mathbb{P}_3 \) lie on some quadric.

Exercise 2.9. Show that any 3 lines in \( \mathbb{P}_3 \) lie on a quadric.
A quadratic surface of rank 1 is called a double plane. It is totally singular and has the equation \( x_0^2 = 0 \) in appropriate coordinates on \( \mathbb{P}_3 \). A quadratic surface \( S \) of rang 2 either is a split quadric, i.e., a union of two planes intersecting along the singular line \( \ell = \text{Sing} S \), or is exhausted by the singular line, and the latter case is impossible over an algebraically closed field.

Exercise 2.10. Prove this.
A quadratic surface \( S \subset \mathbb{P}_3 \) of rank 3 is called a simple cone. It is ruled by the lines \((sp)\), where \( s \in S \) is the singular point and \( t \) runs through a smooth conic \( C = S \cap \Pi \) laying in a plane \( \Pi \not\ni s \). Note that \( C \) may be empty as soon the ground field is not algebraically closed. In this case \( S = \{ s \} \) is exhausted by the singular point. If \( C \neq \emptyset \), the linear span of \( C \) is the whole \( \Pi \).

**Exercise 2.11.** Convince yourself that the lines laying on a simple cone with vertex \( s \) over a smooth conic \( C \) are exhausted by the lines \((st)\), \( t \in C \).

As a byproduct of the previous discussion, we get

**Proposition 2.8**

Every 3 mutually non-intersecting lines in \( \mathbb{P}_3 \) lie on a smooth quadratic surface. \( \square \)

Over an algebraically closed field, all smooth quadrics in \( \mathbb{P}_3 \) are congruent modulo the linear projective automorphisms of \( \mathbb{P}_3 \). The most convenient model of the smooth quadric is described below.

2.5.1 **The Segre quadric.** Let \( U \) be a vector space of dimension 2. Write \( W = \text{End}(U) \) for the space of linear maps \( F : U \to U \), and consider \( \mathbb{P}_3 = \mathbb{P}(W) \). A choice of basis in \( U \) identifies \( W \) with the space \( \text{Mat}_2(\mathbb{K}) \) of \( 2 \times 2 \) matrices. The quadric

\[
S \equiv \{ F : \in \text{End}(U) \mid \det F = 0 \} = \left\{ \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \mid x_0x_3 - x_1x_2 = 0 \right\} \subset \mathbb{P}_3
\]

(2.3)

is called the Segre quadric. It is formed by endomorphisms of rank 1 considered up to proportionality. The image of an operator \( F \) of \( U \to U \) of rank 1 has dimension 1 and is spanned by a non-zero vector \( v \in U \), uniquely determined by \( F \) up to proportionality. The value of \( F \) on an arbitrary vector \( u \in U \) equals \( F(u) = \xi(u) \cdot v \), where \( \xi \in U^* \) is a linear form such that \( \text{Ann} \xi = \ker F \). Note that \( \xi \) is uniquely determined by \( F \) and \( v \in \text{im} F \setminus \{0\} \). Conversely, for any non-zero \( v \in U \), \( \xi \in U^* \) the operator

\[
\xi \otimes v : U \to U, \quad u \mapsto \xi(u)v
\]

has rank 1, its image is spanned by \( v \), and the kernel equals \( \text{Ann} \xi \). Thus, we have the well-defined injective map

\[
s : \mathbb{P}(U^*) \times \mathbb{P}(U) \hookrightarrow \mathbb{P} \text{End}(U), \quad (\xi, v) \mapsto \xi \otimes v .
\]

(2.4)

whose image coincides with the Segre quadric (2.3). This map is called the Segre embedding.

The rows of any \( 2 \times 2 \) matrix of rank 1 are proportional, as well as the columns. The matrices with a fixed ratio \((\text{row 1}) : (\text{row 2}) = (t_0 : t_1) \) or \((\text{column 1}) : (\text{column 2}) = (\xi_0 : \xi_1) \) form a vector subspace of dimension 2 in \( W = \text{Mat}_2(\mathbb{K}) \). After the projectivization these subspaces turn to the two families of lines ruling the Segre quadric. These lines are the images of «coordinate lines» \( \mathbb{P}_1^2 \times \mathbb{P}_1 \) and \( \mathbb{P}_1 \times \mathbb{P}_1 \) on the product \( \mathbb{P}_1^2 \times \mathbb{P}_1 = \mathbb{P}(U^*) \times \mathbb{P}(U) \) under the bijection \( \mathbb{P}_1^2 \times \mathbb{P}_1 \to S \) provided by the Segre embedding (2.4). Indeed, the operator \( \xi \otimes v \) build from from \( \xi = (\xi_0 : \xi_1) \in U^* \) and \( v = (t_0 : t_1) \in U \) has the matrix

\[
\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \cdot (\xi_0 : \xi_1) = \begin{pmatrix} \xi_0t_0 & \xi_1t_0 \\ \xi_0t_1 & \xi_1t_1 \end{pmatrix}
\]

(2.5)

with the prescribed ratios \( t_0 : t_1 \) and \( \xi_0 : \xi_1 \) between the rows and columns respectively. Since the Segre map \( \mathbb{P}_1^2 \times \mathbb{P}_1 \to S \) is bijective, the incidence relations among coordinate lines in \( \mathbb{P}_1^2 \times \mathbb{P}_1 \) are the same as among their images in \( S \). That is, within each ruling family, all the lines
are mutually non-intersecting, every two lines from different ruling families are intersecting, and each point on the Segre quadric is an intersection point of exactly two lines from different families.

**Exercise 2.12.** Prove that all lines \( \ell \subset S \) are exhausted by these two ruling families.

**Proposition 2.9 (continuation of the Proposition 2.8)**

A smooth quadric \( Q \) passing through a triple \( \ell_1, \ell_2, \ell_3 \) of mutually non-intersecting lines in \( \mathbb{P}_3 \), as in the Proposition 2.8, is ruled by all those lines in \( \mathbb{P}_3 \) that do intersect all the lines \( \ell_i \). In particular, this quadric is unique.

**Proof.** If a line \( \ell \) intersects all the lines \( \ell_i \), it has at least 3 distinct points on \( Q \) and therefore lies on \( Q \). On the other side, for any point \( a \in Q \) not laying on the lines \( \ell_i \), the tangent plane \( T_aQ \) intersects every line \( \ell_i \) at some point \( p_i \neq a \). Since the line \((ap_i)\) touches \( Q \) at \( a \), it lies on \( Q \). Thus, all three lines \((ap_i)\) lie on the conic \( Q \cap T_aQ \). Hence, at least two of them, say \((ap_1), (ap_2)\), coincide. If \( p_3 \) does not belong to the line \( \ell = (ap_1) = (ap_2) \), then the tangent plane \( T_{p_3}Q \) intersects \( \ell \) at a point \( b \) different from \( a \) and all \( p_i \)’s. The line \((p_3b) \subset Q \) by the same reason as above. Thus, \( Q \) contains the triangle \( ap_3p_3 \) formed by 3 distinct lines \( \ell, (ap_3), (ab) \). Hence, \( Q \) contains the whole plane spanned by this triangle.

**Exercise 2.13.** Show that a smooth quadric in \( \mathbb{P}_3 \) can not contain a plane.

Therefore, the points \( a, p_1, p_2, p_3 \) are collinear, that is, \( a \) lies on a line intersecting all the lines \( \ell_i \).

**Exercise 2.14.** Given 4 mutually non-intersecting lines in \( \mathbb{P}_3 \), how many lines intersect them all?

### 2.6 Linear subspaces lying on a smooth quadric

A smooth quadric \( Q \) is called \( k \)-planar, if there is a projective subspace \( L \subset Q \) of dimension \( \dim L = k \) and \( Q \) does not contain a subspace of higher dimension. By the definition, the planarity of the empty quadric is \(-1\). Thus, the quadrics of planarity 0 are non-empty and do not contain lines.

**Proposition 2.10**

The planarity of a smooth quadric \( Q \subset \mathbb{P}_n \) does not exceed \( \dim Q / 2 = (n - 1) / 2 \).

**Proof.** Let \( \mathbb{P}_n = \mathbb{P}(V) \) and \( L = \mathbb{P}(W) \subset Q = V(q) \) for some non-singular quadratic form \( q \in S^2V^* \) and a vector subspace \( W \subset V \). Since \( q|_W = 0 \), the correlation \( \hat{q} : V \leftrightarrow V^* \) sends \( W \) into \( \text{Ann}(W) \). Since \( \hat{q} \) is injective, \( \dim(W) = \dim(\hat{q}(W)) \leq \dim(\text{Ann}W) = \dim V - \dim W \). Thus, \( 2 \dim W \leq \dim V \) and \( 2 \dim L \leq n - 1 \).

**Lemma 2.4**

For any smooth quadric \( Q \) and hyperplane \( \Pi \), the intersection \( \Pi \cap Q \) either is a smooth quadric in \( \Pi \) or has exactly one singular point \( p \in \Pi \cap Q \). The latter happens if and only if \( \Pi = T_pQ \).

**Proof.** Let \( Q = V(q) \subset \mathbb{P}(V), \Pi = \mathbb{P}(W) \). Since \( \dim(\ker(\hat{q}|_W)) = \dim(\{W \cap \hat{q}^{-1}(\text{Ann}W)\}) \leq \dim(\hat{q}^{-1}(\text{Ann}W)) = \dim(\text{Ann}W) = \dim V - \dim W = 1 \), the quadric \( \Pi \cap Q \subset \Pi \) has at most one singular point. If \( \text{Sing} Q = \{p\} \neq \emptyset \), then the kernel \( \ker(\hat{q}|_W) \subset W \) has dimension 1 and is spanned by \( p \). Thus, \( \text{Ann}(\hat{q}(p)) = W \), that is, \( T_pQ = \Pi \). Vice versa, if \( \Pi = T_pQ = \mathbb{P}(\text{Ann}(\hat{q}(p))) \), then \( p \in \text{Ann}(\hat{q}(p)) \) belongs to the kernel of the restriction of \( \hat{q} \) on \( \text{Ann}(\hat{q}) \).

\(^1\)Because for every point of the plane except for the vertexes of triangle, every line passing through this point intersects all three lines \( \ell, (ap_3), (ab) \).
PROPOSITION 2.11
Let $Q \subset \mathbb{P}_{n+1}$ be a smooth quadric of dimension $n$. For every $1 \leq m \leq n/2$, the projective subspaces of dimension $m$ lying in $Q$ and passing through a given point $p \in Q$ stay in bijection with all projective subspaces of dimension $m-1$ lying on a smooth quadric of dimension $n-2$ cut out of $Q$ by any hyperplane $H \subset T_pQ$ complementary to $p$ within the tangent hyperplane $T_pQ \cong \mathbb{P}_{n-1}$.

PROOF. Every projective subspace $L \subset Q$ of dimension $m$ passing through $p \in Q$ lies inside the intersection $Q \cap T_pQ$, which is the singular quadric in $\mathbb{P}_{n-1} = T_pQ$ with just one singular point $p$ by the Lemma 2.4. It accordance with the Theorem 2.1 on p. 18, the quadric $Q \cap T_pQ \subset \mathbb{P}_{n-1}$ is the cone ruled by lines $(\alpha a)$, where $\alpha$ runs through the smooth quadric $Q'$ cut out of $Q$ by a hyperplane $H \subset \mathbb{P}_{n-1}$ not passing through $p$. Thus, the subspaces $L \subset Q \cap T_pQ$ of dimension $n$ are exactly the linear joins of $p$ with the subspaces $L' = L \cap H = L \cap Q'$ of dimension $m-1$ laying on $Q'$. □

COROLLARY 2.5
For any two distinct points $a$, $b$ on a smooth quadric $Q$ and all $0 \leq m \leq \dim Q/2$ there is a bijection between the subspaces of dimension $m$ lying on $Q$ and passing through the points $a$ and $b$ respectively. In particular, a projective subspace of dimension $k$ laying on a smooth $k$-planar quadric can be drawn through every point of the quadric.

PROOF. If $b \notin T_aQ$, then $H = T_aQ \cap T_bQ$ does not pass through $a$, $b$ and lies in the both tangent spaces $T_aQ$, $T_bQ$ as a hyperplane. By the Proposition 2.11, the sets of projective subspaces $L \subset Q$ of dimension $m$ passing through $a$ and $b$ respectively both stay in bijection with the subspaces $L' \subset Q \cap H$ of dimension $m-1$. If $b \in T_aQ$, pick up a point $c \in Q \setminus (T_aQ \cup T_bQ)$ and repeat the previous arguments twice for the pairs $a$, $c$ and $c$, $b$. □

COROLLARY 2.6
A smooth quadric of dimension $n$ over an algebraically closed field is $[n/2]$-planar.

PROOF. This holds for $n = 0, 1, 2$. Then we use the Proposition 2.11 and induction in $n$. □
Comments to some exercises

Exrc. 2.4. This follows from the last representation from formula (2.1) on p. 16.

Exrc. 2.5. Let \( \mathbb{P}(\text{Ann } \xi) \cup \mathbb{P}(\text{Ann } \eta) \) for some non zero covectors \( \xi, \eta \in V^* \). Then the quadratic form \( q(v) = \xi(v)\eta(v) \) vanishes identically on \( V \). Therefore its polarization \( \tilde{q}(u, w) = (q(u + w) - q(u) - q(w))/2 \) also vanishes. Hence, the Gram matrix of \( q \) equals zero, i.e., \( q \) is the zero polynomial. However, the polynomial ring has no zero divisors.

Exrc. 2.7. Use the Lemma 2.1 on p. 18 and prove that non-empty smooth quadric over an infinite field can not be covered by a finite number of hyperplanes.

Exrc. 2.9. Pick up some \( \mathbb{Q} \) on each line and draw a quadric through these 9 points.

Exrc. 2.10. By the Theorem 2.1 on p. 18, \( S \) is the linear join of the singular line \( \text{Sing} S \) and a smooth quadric \( S \cap \ell \) within a line \( \ell \) complementary to \( \text{Sing} S \). This smooth quadric is either a pair of distinct points or empty.

Exrc. 2.12. Every line \( \ell \subset S \) passing through a given point \( a \in S \) lies inside \( S \cap T_a S \), which is the split conic exhausted by two ruling lines crossing at \( a \).

Exrc. 2.13. See the Proposition 2.10 on p. 23.

Exrc. 2.14. Use the method of loci: remove one of the given lines and look how does the locus filled by the lines crossing 3 remaining lines interact with the removed line.