§3 Working examples: lines and conics on the plane

3.1 Homographies. A linear projective isomorphism between two projective lines is called *a homography*. An important example of homography is provided by *a perspective* $o : \ell_1 \xrightarrow{\sim} \ell_2$, the central projection of a line $\ell_1 \subset \mathbb{P}_2$ to another line $\ell_2 \subset \mathbb{P}_2$ from a point $o \notin \ell_1 \cup \ell_2$, see fig. 3⋄1.

EXERCISE 3.1. Make sure that a perspective is a homography.

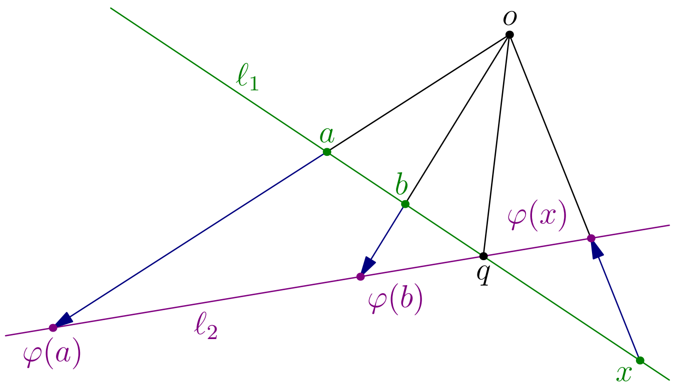

Fig. 3⋄1. The perspective $o : \ell_1 \xrightarrow{\sim} \ell_2$.

A homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ is a perspective if and only if it sends the intersection point $\ell_1 \cap \ell_2$ to itself. Indeed, choose two distinct points $a, b \in \ell_1 \smallsetminus \ell_2$ and put $o = (a\varphi(a)) \cap (b\varphi(b))$ as on fig. 3⋄1. Then the perspective $o : \ell_1 \xrightarrow{\sim} \ell_2$ sends the points a, b, $\ell_1 \cap \ell_2$ to $\varphi(a)$, $\varphi(b)$, $\ell_1 \cap \ell_2$. Thus, it coincides with φ if and only if φ maps the intersection of lines to itself.

3.1.1 The cross-axis. Given two lines $\ell_1, \ell_2 \subset \mathbb{P}_2$ intersecting at the point $q = \ell_1 \cap \ell_2$, then for any line $\ell \subset \mathbb{P}_2$ and points $b_1 \in \ell_1$, $b_2 \in \ell_2$ the composition of perspectives

$$(b_1 : \ell \to \ell_2) \circ (b_2 : \ell_1 \to \ell) \tag{3-1}$$

takes $b_1 \mapsto b_2$, $\ell_1 \cap \ell \mapsto q$, $q \mapsto \ell_2 \cap \ell$, see fig. 3⋄2.

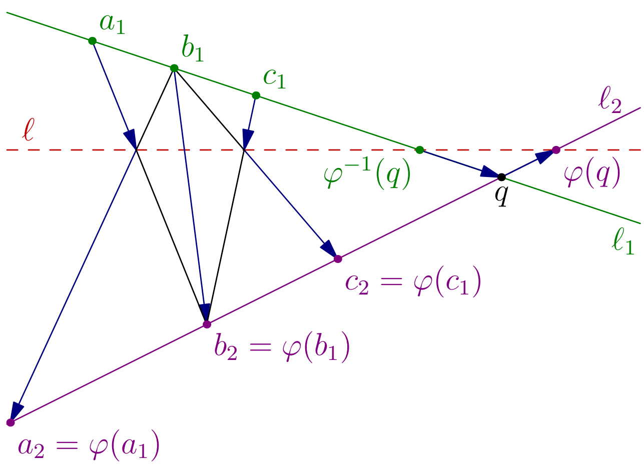

Fig. 3⋄2. The cross-axis of a homography.

Every homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ admits a decomposition (3-1) in which the point $b_1 \in \ell_1$ can be chosen arbitrarily, $b_2 = \varphi(b_1)$, and the line ℓ is uniquely predicted by φ and does not depend on the choice of $b_1 \in \ell_1$. Indeed, fix some distinct points $a_1, b_1, c_1 \in \ell_1 \smallsetminus \ell_2$ and write $a_2, b_2, c_2 \in \ell_2$ for their images under φ. Put ℓ as the line joining the cross-intersections $(a_1 b_2) \cap (b_1 a_2)$ and $(c_1 b_2) \cap (b_1 c_2)$. Then the composition (3-1) sends a_1, b_1, c_1 to a_2, b_2, c_2 and therefore coincides with φ, see fig. 3◇2. If we repeat this argument for the ordered triple c_1, a_1, b_1 instead of a_1, b_1, c_1, then we get the decomposition $\varphi = (a_1 : \ell' \to \ell_2) \circ (a_2 : \ell' \to \ell)$, where ℓ' joins the cross-intersections $(a_1 c_2) \cap (c_1 a_2)$ and $(b_1 a_2) \cap (a_1, b_2)$, see fig. 3◇3. Since both lines ℓ, ℓ' pass through the points[1] $(b_1 a_2) \cap (a_1, b_2)$, $\varphi(q)$, $\varphi^{-1}(q)$, we conclude that $\ell = \ell'$. Hence, all the cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$, where $x \neq y$ are running through ℓ_1, lie on the same line ℓ, which is uniquely determined by this property.

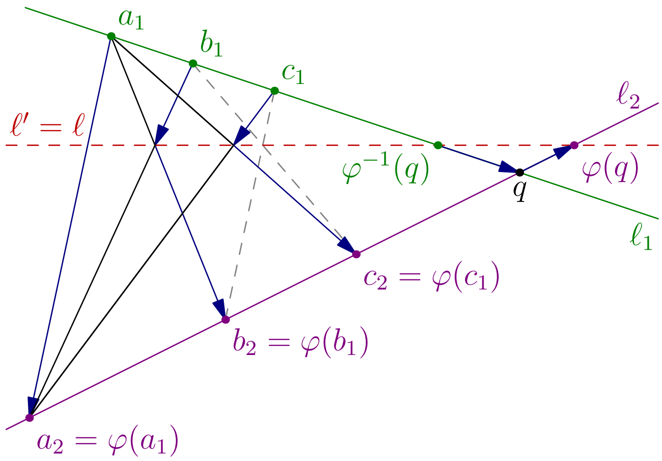

Fig. 3◇3. Coincidence $\ell' = \ell$.

DEFINITION 3.1 (THE CROSS-AXIS OF HOMOGRAPHY)
Given a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$, the line ℓ drown by cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$ as $x \neq y$ run through ℓ_1 is called the *cross-axis* of φ.

REMARK 3.1. The cross-axis of non-perspective homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ is well defined as the line joining $\varphi(\ell_1 \cap \ell_2)$ and $\varphi^{-1}(\ell_1 \cap \ell_2)$, which are distinct. If φ is a perspective, then the point $\varphi(\ell_1 \cap \ell_2) = \varphi^{-1}(\ell_1 \cap \ell_2) = \ell_1 \cap \ell_2$ still lies on the cross-axis but does not fix it uniquely.

EXERCISE 3.2. Let a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ send 3 given points $a_1, b_1, c_1 \in \ell_1$ to 3 given points $a_2, b_2, c_2 \in \ell_2$. Using only the ruler, construct $\varphi(x)$ for a given $x \in \ell_1$.

LEMMA 3.1
Let $\Bbbk$ be an algebraically closed field of zero characteristic. If a bijection

$$\varphi : \mathbb{P}_1(\Bbbk) \smallsetminus \{\text{finite set of points}\} \xrightarrow{\sim} \mathbb{P}_1(\Bbbk) \smallsetminus \{\text{finite set of points}\}$$

can be described in some affine chart with a local coordinate t by a formula

$$\varphi : t \mapsto \varphi_0(t)/\varphi_1(t), \quad \text{where } \varphi_0, \varphi_1 \in \Bbbk[t], \tag{3-2}$$

then φ is the restriction of a unique homography $\mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$.

[1] Note that the latter two coincide as soon φ is a perspective.

Proof. In the homogeneous coordinates $(x_0 : x_1)$ such that $t = x_0 / x_1$, the formula (3-2) can be rewritten[1] as $\varphi : (x_0 : x_1) \mapsto \big(f_0(x_0, x_1) : f_1(x_0, x_1)\big)$, where $f_0, f_1 \in \Bbbk[x_0, x_1]$ are non-proportional homogeneous polynomials of the same degree d. Write $\mathbb{P}_d$ for the projectivization of space of homogeneous polynomials of degree d in x_0, x_1. As soon as a point $\vartheta = (\vartheta_0 : \vartheta_1) \in \mathbb{P}_1$ has a unique preimage under φ, the polynomial $h_\vartheta(x_0, x_1) = \vartheta_1 f(x_0, x_1) - \vartheta_0 g(x_0, x_1)$ has just one root in $\mathbb{P}_1$. Since $\Bbbk$ is algebraically closed, h_ϑ is the proper d th power of a linear form, that is, lies on the Veronese curve[2] $C_d \subset \mathbb{P}_d$. On the other hand, the polynomial h_ϑ runs through the line $(f_0, f_1) \subset \mathbb{P}_d$ as ϑ runs through $\mathbb{P}_1$. Since $\mathbb{P}_1(\Bbbk)$ is infinite, we conclude that the Veronese curve has infinitely many intersections with the line (f_0, f_1). But for $d \geqslant 2$, any 3 distinct points of C_d are non-collinear[3]. Hence, $d = 1$ and $\varphi \in \mathrm{PGL}_2(\Bbbk)$. □

3.1.2 Homographies provided by conics. Let a homography $\varphi : \ell_1 \simeq \ell_2$ send an ordered triple of distinct points $a_1, b_1, c_1 \in \ell_1 \smallsetminus \ell_2$ to $a_2, b_2, c_2 \in \ell_2$. If the lines $(a_1 a_2)$, $(b_1 b_2)$, $(c_1 c_2)$ meet all together at some point p, then φ coincides with the perspective $p : \ell_1 \simeq \ell_2$, and this happens if and only if $\varphi(q) = q$, see fig. 3◇4.

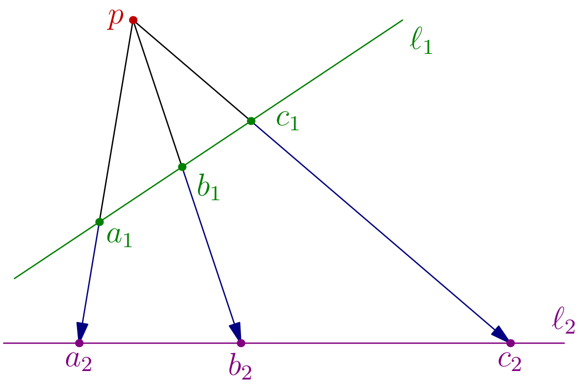

Fig. 3◇4. Perspective $p : \ell_1 \to \ell_2$.

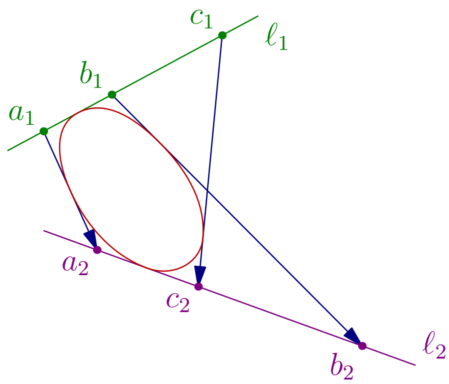

Fig. 3◇5. Homography $C : \ell_1 \to \ell_2$.

If the lines $(a_1 a_2)$, $(b_1 b_2)$, $(c_1 c_2)$ are not concurrent, then any 3 of the 5 lines ℓ_1, ℓ_2, (a_1, a_2), (b_1, b_2), (c_1, c_2) are not concurrent, and there exists a unique smooth conic C touching all these 5 lines by the Corollary 2.4 on p. 21, see fig. 3◇5. In this case, the homography φ is provided by the tangent lines to C, i.e., $y = \varphi(x)$ if and only if the line (xy) is tangent to C. Indeed, the map $C : \ell_1 \to \ell_2$, which sends $x \in \ell_1$ to the intersection point of ℓ_2 with the tangent line from x to C other than ℓ_1, is obviously bijective.

Exercise 3.3. Convince yourself that this map satisfies the Lemma 3.1.

We conclude that $C : \ell_1 \to \ell_2$ is a homography that acts on a_1, b_1, c_1 exactly as φ.

Thus, every homography $\varphi : \ell_1 \simeq \ell_2$ is either a perspective $p : \ell_1 \simeq \ell_2$ provided by some point $p \notin \ell_1 \cup \ell_2$ or a homography $C : \ell_1 \to \ell_2$ provided by a smooth conic C touching the both lines ℓ_1, ℓ_2. In both cases, the point p and conic C are uniquely predicted by φ. The perspective $p : \ell_1 \simeq \ell_2$ can be treated as a degeneration of the non-perspective homography $C : \ell_1 \simeq \ell_2$ arising when C splits in two lines crossing at the centre of perspective. However these two lines can

[1] Perhaps, after a modification of the finite set on which φ is undefined.
[2] See n° 1.3.3 on p. 9.
[3] See n° 1.3.3 on p. 9.

be chosen in many ways: any two lines joining the corresponding points are fitted in the picture. Note also that the image and preimage of $\ell_1 \cap \ell_2$ under the homography $C : \ell_1 \simeq \ell_2$ are the points of contact $\ell_2 \cap C$ and $\ell_1 \cap C$ respectively.

Proposition 3.1 (inscribed-circumscribed triangles)
Two triangles $\triangle\, a_1 b_1 c_1$ and $\triangle\, a_2 b_2 c_2$ are both inscribed in some smooth conic Q' if and only if they are both circumscribed about some smooth conic Q''.

Proof. Let 6 points $a_1, b_1, c_1, a_2, b_2, c_2$ lie on a smooth conic Q' like in fig. 3⋄6. Put $\ell_1 = (a_1 b_1)$, $\ell_2 = (a_2 b_2)$ and write $c_2 : \ell_1 \simeq Q'$ for the projection of ℓ_1 onto Q' from c_1 and $c_1 : Q' \simeq \ell_2$ for the projection of Q' onto ℓ_2 from c_2. The composition $[c_1 : Q' \simeq \ell_2] \circ [c_2 : \ell_1 \simeq Q'] : \ell_1 \simeq \ell_2$ is a non-perspective homography sending $a_1 \mapsto p$, $q \mapsto b_2$, $r \mapsto a_2$, $b_1 \mapsto s$. Let Q'' be a smooth conic whose tangent lines join the homographic points. Then Q'' is obviously inscribed in the both triangles. The opposite implication is projectively dual to just proven. □

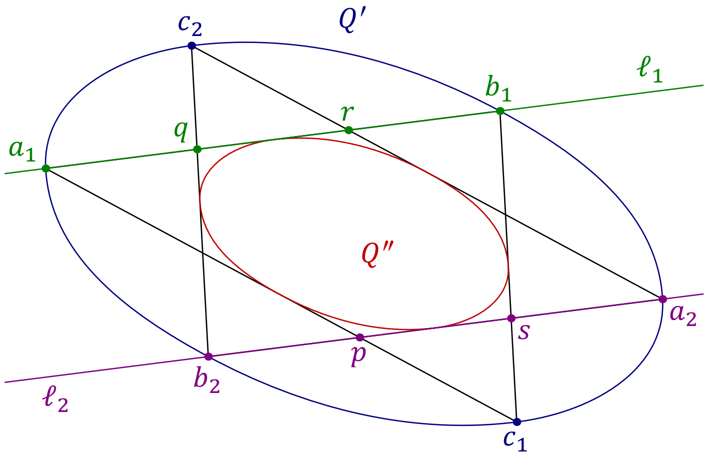

Fig. 3⋄6. Inscribed circumscribed triangles.

Corollary 3.1 (Poncelet's porism for triangles)
Assume that a triangle $\triangle\, a_1 b_1 c_1$ is simultaneously inscribed in a smooth conic Q' and circumscribed about a smooth conic Q''. Then every point of Q' except for a finite set is a vertex of triangle simultaneously inscribed in Q' and circumscribed about Q''.

Proof (see fig. 3⋄6). For any $a_2, b_2, c_2 \in Q'$ such that $(a_2 b_2)$, $(a_2 c_2)$ are two different tangent lines to Q'', the triangles $\triangle\, a_1 b_1 c_1$ and $\triangle\, a_2 b_2 c_2$ are both circumscribed about some smooth conic C by the Proposition 3.1. Since C touches 5 lines $(a_1 b_1)$, $(b_1 c_1)$, $(c_1 a_1)$, $(a_2 b_2)$, $(a_2 c_2)$, it coincides with Q'' by the Corollary 2.4 on p. 21. □

3.1.3 Homographic pencils of lines. Projectively dual version of the construction from n° 3.1.2 deals with a homography $\varphi : p_1^\times \simeq p_2^2$ between two pencils of lines in $\mathbb{P}_2$ passing through the points p_1 and p_2 respectively. Let φ sent 3 distinct lines $\ell_1', \ell_2', \ell_3' \ni p_1$ other than $(p_1 p_2)$ to the lines $\ell_1'', \ell_2'', \ell_3'' \ni p_1$. Write $q_i = \ell_i' \cap \ell_i''$, $i = 1, 2, 3$, for the intersection points of corresponding lines. Since every 4 points from p_1, p_2, q_1, q_2, q_3 are non-collinear, there exists the unique conic

C_φ passing through these 5 points, see fig. 3⋄7 and fig. 3⋄8 below. Provided by this conic is the homography $C : p_1^\times \simeq p_2^\times$ sending $(p_1 p) \mapsto (p_2 p)$ for all $p \in C_\varphi$.

EXERCISE 3.4. Use the Lemma 3.1 on p. 26 to convince yourself that this map is actually a homography.

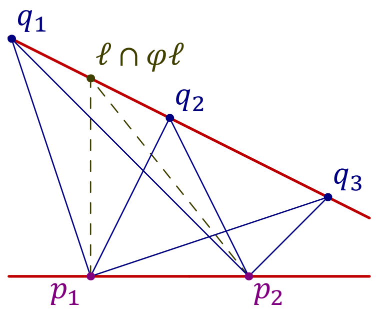

Fig. 3⋄7. Perspective homography $\varphi : p_1^\times \to p_2^\times$.

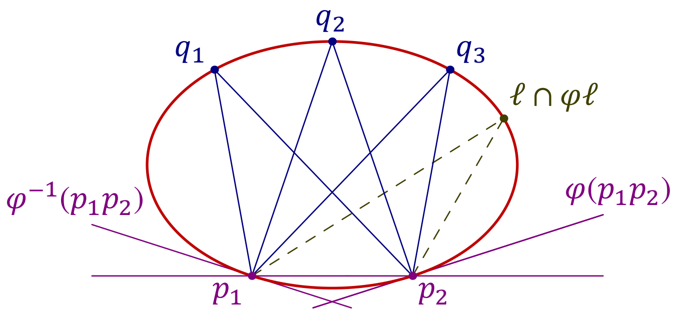

Fig. 3⋄8. Non-perspective homography $\varphi : p_1^\times \to p_2^\times$.

Since this homography takes $\ell_i' \mapsto \ell_i''$ for $i = 1, 2, 3$, it coincides with φ, see. fig. 3⋄8. The homography provided by a smooth conic C_φ takes $T_{p_1}C_\varphi \mapsto (p_1p_2)$ and $(p_1p_2) \mapsto T_{p_2}C_\varphi$. The conic C_φ splits if and only if the points q_1, q_2, q_3 are collinear or, equivalently, when the line (p_1p_2) goes to itself. In this case $C_\varphi = (p_1p_2) \cup (q_iq_j)$ and the homography is a perspective, see fig. 3⋄7. In a contrast with n° 3.1.2, the split conic C_φ is uniquely determined by the perspective φ in this case.

EXAMPLE 3.1 (TRACING CONIC BY THE RULER)
Let C be a conic drawn through 5 given points $p_1, p_2, \dots, p_5$ no 3 of which are collinear. The points of C can be constructed by the ruler as follows. Draw the lines $\ell_1 = (p_2p_5)$, $\ell_2 = (p_2p_4)$ and mark the point $p = (p_1p_4) \cap (p_3p_5)$, see fig. 3⋄9.

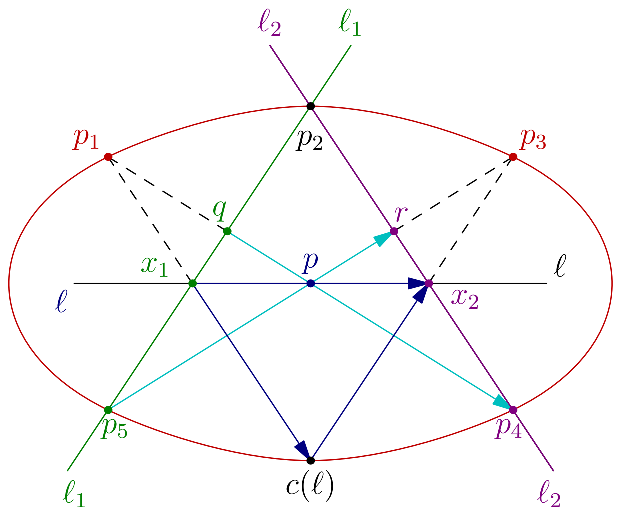

Fig. 3⋄9. Tracing a conic by a ruler.

The perspective $p : \ell_1 \simeq \ell_2$ is decomposed as the projection $p_1 : \ell_1 \simeq C$ of ℓ_1 onto C from p_1 followed by projection $p_3 : C \simeq \ell_2$ from C onto ℓ_2 from p_3.

EXERCISE 3.5. Check this by comparing the action on points $p_2, p_5, q \in \ell_1$, see fig. 3⋄9.

Thus, for any line $\ell \ni p$, the lines joining p_1, p_2 with the intersection points $x_1 = \ell \cap \ell_1$, $x_2 = \ell \cap \ell_2$ are crossing at the point $c(\ell) = (p_1 x_1) \cap (p_2 x_2) \in C$, see fig. 3⋄9. As ℓ turns about p, the point $c(\ell)$ draws the conic C.

Theorem 3.1 (Pascal's theorem)
Six points $p_1, p_2, \dots, p_6$ no 3 of which are collinear lie on a smooth conic if and only if 3 intersection points[1] $x = (p_3 p_4) \cap (p_6 p_1)$, $y = (p_1 p_2) \cap (p_4 p_5)$, $z = (p_2 p_3) \cap (p_5 p_6)$ are collinear.

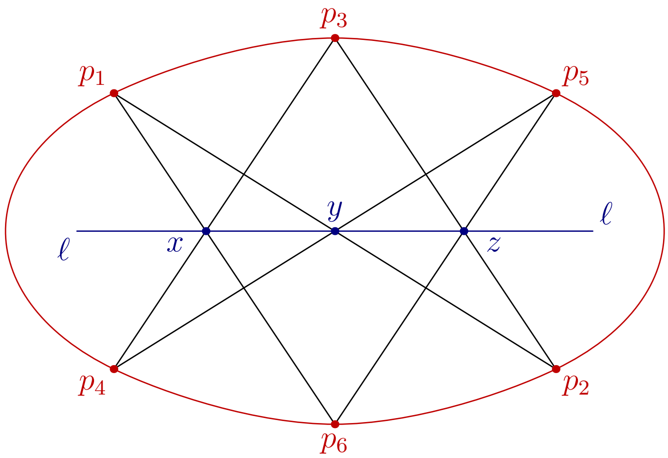

Fig. 3⋄10. The hexogram of Pascal.

Proof. Let $\ell_1 = (p_3 p_4)$, $\ell_2 = (p_3 p_2)$, see fig. 3⋄10. Assume that $z \in (xy)$. Then the perspective $y : \ell_1 \to \ell_2$ takes $x \mapsto z$ and is decomposed[2] as $\left(p_5 : C \simeq \ell_2\right) \circ (p_1 : \ell_1 \simeq C)$, where C is the smooth conic passing trough $p_1, p_2, \dots, p_5$. Thus, $p_6 = (p_5 z) \cap (p_3 x) \in C$. Conversely, if $(p_5 z) \cap (p_3 x) \in C$, then the above composition takes $x \mapsto z$. Hence, the perspective $y : \ell_1 \to \ell_2$ also sends $x \mapsto z$ forcing $z \in (xy)$. □

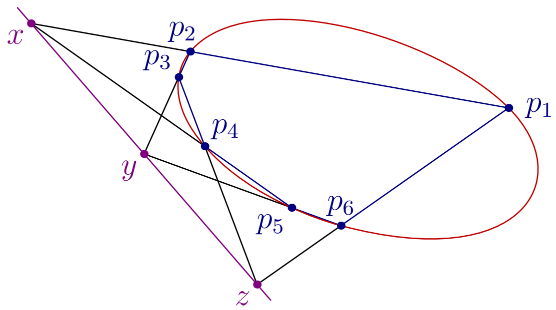

Fig. 3⋄11. Inscribed hexagon.

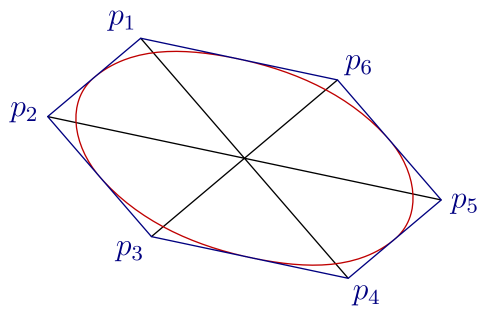

Fig. 3⋄12. Circumscribed hexagon.

Corollary 3.2 (Brianchon's theorem)
A hexagon $p_1, p_2, \dots, p_6$ is circumscribed about a non-singular conic if and only if «the main diagonals» $(p_1 p_4)$, $(p_2 p_5)$, $(p_3 p_6)$ are concurrent, see fig. 3⋄12.

Proof. This is dual to the Theorem 3.1, comp. fig. 3⋄11 and fig. 3⋄12. □

[1] They can be thought of as intersection points of «the opposite sides» of hexagon $p_1, p_2, \dots, p_6$.
[2] See the Example 3.1 on p. 29.

3.2 Internal geometry of a smooth conic. In this section we assume on default that the ground field $\Bbbk$ is algebraically closed and $\mathrm{char}(\Bbbk) \neq 2$. Dual projective lines $\mathbb{P}_1 = \mathbb{P}(U)$, $\mathbb{P}_1^{\times} = \mathbb{P}(U^*)$ are naturally identified by the canonical homography provided by projective duality:

$$\delta : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1^{\times}, \quad v \mapsto \mathrm{Ann}\, v. \tag{3-3}$$

In coordinates, it takes a point $(p_0 : p_1) \in \mathbb{P}_1$ to the linear form $\det(p, t) = p_0 t_1 - p_1 t_0$, whose coordinates in the dual basis of $\mathbb{P}_1^{\times}$ are $(-p_1 : p_0)$. The plane $\mathbb{P}_2 = \mathbb{P}(S^2 U^*)$ can be thought[1] of as the space of non-ordered pairs of possibly coinciding points in $\mathbb{P}_1 = \mathbb{P}(U)$ by mapping a pair of points $p = (p_0 : p_1)$, $q = (q_0 : q_1)$ on $\mathbb{P}_1$ to the binary quadratic form with roots $\{p, q\}$:

$$\begin{aligned} f_{pq}(t_0, t_1) = \det\begin{pmatrix} p_0 & t_0 \\ p_1 & t_1 \end{pmatrix} \det\begin{pmatrix} q_0 & t_0 \\ q_1 & t_1 \end{pmatrix} = \\ = p_0 q_0 \cdot t_0^2 - (p_0 q_1 + p_1 q_0) \cdot t_0 t_1 + p_1 q_1 \cdot t_1^2 \in S^2 U^*. \end{aligned} \tag{3-4}$$

We will often misuse the notations and write $\{p, q\} \in \mathbb{P}_2$ for the quadratic form (3-4). Pairs $\{p, t\} \in \mathbb{P}_2$, where $p \in \mathbb{P}_1$ is fixed and t runs through $\mathbb{P}_1$, form a line in $\mathbb{P}_2$. This line consists of all $f \in S^2(U^*)$ such that $f(p) = 0$. Pairs of coinciding points $\{p, p\} \in \mathbb{P}_2$ form the smooth Veronese conic $C \subset \mathbb{P}_2$. The above line $\{p, t\}$ is tangent to C at the point $\{p, p\}$, certainly. Thus, the pair of tangent lines to C drown through a point $\{p, q\} \notin C$ is formed by $\{p, t\}$, $\{q, t\}$, where $t \in \mathbb{P}_1$, which meet C at the points $\{p, p\}$, $\{q, q\}$.

The Veronese conic stays in the natural bijection with $\mathbb{P}_1$ provided by the Veronese map[2]

$$\mathbb{P}_1 \hookrightarrow \mathbb{P}_2, \quad p \mapsto \{p, p\}.$$

In coordinates, it takes a point $(p_0 : p_1) \in \mathbb{P}_1$ to the binary quadratic form $x_0 t_0^2 + 2x_1 t_0 t_1 + x_2 t_2^2$ with coefficients

$$(x_0 : x_1 : x_2) = (p_0^2 : -p_0 p_1 : p_1^2). \tag{3-5}$$

We refer the ratio $(p_0 : p_1)$ as *the internal homogeneous coordinate* of the point $\{p, p\}$ on the Veronese conic, and define the cross-ratio of four points $\{p_i, p_i\}$, $i = 1, \dots, 4$, on C as $[p_1, p_2, p_3, p_4]$ on $\mathbb{P}_1$. Note that the internal homogeneous coordinates on C are predicted by a choice of basis in $\mathbb{P}_1$ whereas the cross-ratio does not depend on a choice of coordinates.

As soon as $\Bbbk$ is algebraically closed and $\mathrm{char}\, \Bbbk \neq 2$, every smooth conic D on the plane can be identified with the Veronese conic C by means of linear projective automorphism of the plane. This allows to introduce internal homogeneous coordinates and the cross-ratio on D. We would like to verify that different choices of the linear projective automorphism $\varphi : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ such that $\varphi(D) = C$ do not change the cross-ratio and lead to invertible linear changes of the internal homogeneous coordinates. To this aim, let us redefine the cross-ratio more geometrically.

Definition 3.2 (the cross-ratio on a smooth conic)

Given an ordered quadruple of different points a_1, a_2, a_3, a_4 on a smooth conic D, consider a point $c \in D$ other than given. The cross-ratio of lines $\big[(ca_1), (ca_2), (ca_3), (ca_4)\big]$ in the pencil $c^{\times}$ of lines passing through c is called *the cross-ratio* of points a_i on D.

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[1] See n° 1.3.3 on p. 9.

[2] Note that this map differs from the map $\mathbb{P}_1^{\times} \hookrightarrow \mathbb{P}_2$, described in formula (1-5) on p. 10 and the Example 1.4, by composing with the latter with duality isomorphism $\mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1^{\times}$ from (3-3).

EXERCISE 3.6. Prove that the cross-ratio does not depend on the choice of c and is preserved by linear projective automorphisms of the plane.

Since the parameterization (3-5) of the Veronese conic $C : x_0x_2 = x_1^2$ can be obtained by composing the projection[1] $a : \ell \simeq C$ of the line $\ell : x_2 = 0$ onto C from the point $a = (0 : 0 : 1) \in C$

EXERCISE 3.7. Verify that this projection takes $(p_0 : p_1 : 0) \mapsto (p_0^2 : p_0p_1 : p_1^2)$.

with the homography $\ell \simeq \ell$, $(p_0 : p_1 : 0) \mapsto (p_0 : -p_1 : 0)$, the Definition 3.2 agrees with the previous definition of homogeneous coordinates and cross-ratio on the Veronese conic.

PROPOSITION 3.2
The smooth conic D passing through 5 points $p_1, p_2, \dots, p_5$ no 3 of which are collinear consists of all the points $p \in \mathbb{P}_2$ such that $\big[(pp_1), (pp_2), (pp_3), (pp_4)\big] = \big[(p_5p_1), (p_5p_2), (p_5p_3), (p_5p_4)\big]$.

PROOF. It follows from the Exercise 3.7 that the equality between cross-ratios holds for all points $p \in D$. Consider any point $p \in \mathbb{P}_2$ for which the equality holds, and write Q for the conic passing through p, p_1, p_2, p_3, p_5. Provided by Q is the homography[2] $Q : p^\times \to p_5^\times$ sending a line (pq) to the line (p_5q) for all $q \in Q$. It takes $(pp_i) \mapsto (p_5p_i)$ for $i = 1, 2, 3$. Since $[(pp_1), (pp_2), (pp_3), (pp_4)] = [(p_5p_1), (p_5p_2), (p_5p_3), (p_5p_4)]$, the line (pp_4) goes to the line (p_5p_4). Hence, $p_4 \in Q$ and therefore $Q = D$, because D is the only conic passing through $p_1, p_2, \dots, p_5$. Thus, $p \in D$. □

EXERCISE 3.8. Given 5 points $p, q, a, b, c \in \mathbb{P}_2$ any 3 of which are non-collinear, consider the homography of pencils $\gamma : p^\times \to q^\times$ sending the lines (pa), (pb), (pc) to the lines (qa), (qb), (qc). Describe the locus of intersection points $\ell \cap \gamma(\ell)$ for $\ell \in p^\times$.

3.2.1 Homographies on a smooth conic. A bijection $\varphi : C \simeq C$ provided by an invertible linear change of internal homogeneous coordinates on a smooth conic C is called *a homography*. It follows from the Lemma 3.1 on p. 26 that every rational bijection of the form

$$\varphi : C \smallsetminus \{\text{finite set of points}\} \simeq C \smallsetminus \{\text{finite set of points}\} \tag{3-6}$$

$$(t_0 : t_1) \mapsto \big(f_0(t_0/t_1) : f_1(t_0/t_1)\big), \tag{3-7}$$

where $f_0, f_1 \in \Bbbk[t_0, t_1]$, is the restriction of unique homography $C \simeq C$. For any two ordered triples of distinct points on C there exists a unique homography sending one triple to the other. A bijection $C \simeq C$ is a homography if and only if it preserves the cross-ratio on C.

PROPOSITION 3.3
Every homography $\gamma : C \simeq C$ on a smooth conic $C \subset \mathbb{P}_2$ admits the unique extension to a linear projective automorphism $\widetilde{\gamma} : \mathbb{P}_2 \simeq \mathbb{P}_2$ of the plane. Conversely, any linear projective automorphism $\varphi : \mathbb{P}_2 \simeq \mathbb{P}_2$ such that $\varphi(C) = C$ induces the homography $\varphi|_C : C \simeq C$.

PROOF. Chose 5 distinct points $p_1, p_2, \dots, p_5 \in C$, let $\gamma : C \simeq C$ be a homography, and put $q_i = \gamma(p_i)$. There exists a unique linear projective automorphism $\widetilde{\gamma} : \mathbb{P}_2 \simeq \mathbb{P}_2$ such that $\widetilde{\gamma}(p_i) = q_i$ for $1 \leqslant i \leqslant 4$. Since $\widetilde{\gamma}$ preserves the cross-ratio in the corresponding pencils of lines, the cross-ratio of lines (q_5, q_i), $1 \leqslant i \leqslant 4$, in the pencil $q_5^\times$ equals the cross-ratio of lines (p_5, p_i), $1 \leqslant i \leqslant 4$, in the pencil $p_5^\times$. Since the latter equals the cross-ratio of lines (p_5, q_i), $1 \leqslant i \leqslant 4$, in the same pencil,

[1] See the Example 1.5 on p. 11.
[2] See n° 3.1.3 on p. 28.

because $\gamma : C \xrightarrow{\sim} C$ is the homography and preserves the cross-ratio on C. Thus, for any 5 points $p_1, p_2, \dots, p_5 \in C$ the cross-ratios of lines passing through p_1, p_2, p_3, p_4 in the pencils $p_5^\times$ and $\widetilde{\gamma}(p_5)^\times$ coincide. Hence, $\widetilde{\gamma}(p_5) \in C$ by the Proposition 3.2. The converse statement follows from the Exercise 3.6. □

Example 3.2 (involutions)
A self-inverse homography $\sigma : C \to C$, $\sigma^2 = \mathrm{Id}_C$, is called *an involution* of the conic C. The identity involution $\sigma = \mathrm{Id}_C$ is referred to as *trivial*.

Let an involution $\sigma : C \to C$ interchange a' with a'' and b' with b'' for some mutually different points $a', a'', b', b'' \in C$, as on fig. 3◇13. Consider the intersection point $s = (a'a'') \cap (b'b'')$. Provided by s is the involution $\sigma_s : C \xrightarrow{\sim} C$ swapping the pair of intersection points $\ell \cap C$ on every line $\ell \ni s$.

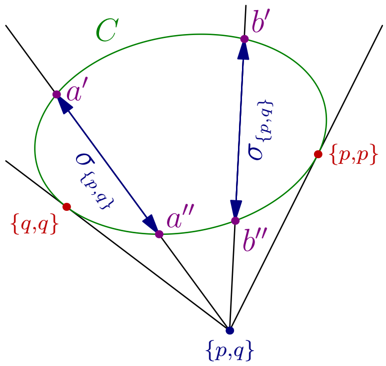

Fig. 3◇13. Involution of conic.

Exercise 3.9. Convince yourself that the map σ_s satisfies the conditions of the Lemma 3.1 on p. 26, and therefore it is a homography.

Since the actions of σ_s and σ on 4 points a', a'', b', b'' coincide, $\sigma = \sigma_s$. In particular, every non-trivial involution has exactly two distinct fixed points[1], the points of contact of two tangent lines to C coming from s. If C is identified with the Veronese conic, the fixed points of involution $\sigma_{p,q}$ are $\{p,p\}$ and $\{q,q\}$. We conclude that every involutive homography $\gamma : \mathbb{P}_1 \to \mathbb{P}_1$ over algebraically closed field has exactly two distinct fixed points $p, q \in \mathbb{P}_1$, and $\gamma(a) = b$ if and only if the points $\{a,a\}$, $\{b,b\}$, $\{p,q\}$ are collinear in $\mathbb{P}_2$.

Exercise 3.10. Verify that the latter is equivalent to the harmonicity $[p,q,a,b] = -1$.

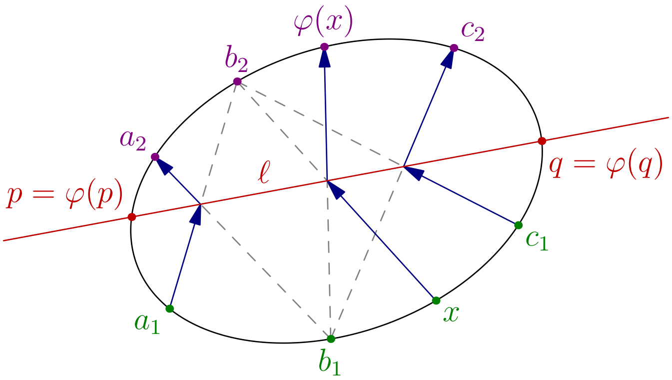

Fig. 3◇14. The cross-axis of a homography on conic.

3.2.2 The cross-axis of a homography on conic.

A homography $\varphi : C \xrightarrow{\sim} C$ sending a_1, b_1, c_1 to $a_2, b_2, c_2 \in C$ is decomposed as projection $b_2 : C \to \ell$ followed by projection $b_1 : \ell \to C$, where ℓ is the line joining cross-intersections $(a_1b_2) \cap (b_1a_2)$ and $(c_1b_2) \cap (b_1c_2)$, see fig. 3◇14. Since the intersection points $\ell \cap C$ are exactly the fixed points[2] of φ, the line ℓ is uniquely predicted by φ

[1] Recall, we assume that $\Bbbk$ is algebraically closed and char $\Bbbk \neq 2$.

[2] In particular, this forces φ to have either two distinct fixed points or just one fixed «double point», and the latter means that ℓ is tangent to C at the fixed point. Note that in both cases ℓ is uniquely recovered from the set of fixed points.

and does not depend on the choice of points $a_1, b_1, c_1 \in C$. In other words, the intersection point of crossing lines $(x, \varphi(y)) \cap (y, \varphi(x))$ draws the line ℓ as $x \neq y$ run through C. This gives another proof for the Pascal theorem[1]: the opposite sides of hexagon $a_1c_2b_1a_2c_1b_2$ inscribed in C are the crossing lines for the homography sending a_1, b_1, c_1 to a_2, b_2, c_2, and therefore their intersection points lie on the cross-axix ℓ of this homography.

The cross axis of a homography $\varphi : C \to C$ can by easily drawn by the ruler as soon the action of φ on some triple of points is known. This allows to construct the image $\varphi(z)$ of any given point $z \in C$, and to find the fixed points of φ using only the ruler. In particular, given a smooth conic C and point s in $\mathbb{P}_2$, it is not hard to draw the tangent lines to C from s by means of the ruler only: one could either construct the fixed points of involution $\sigma_s : C \to C$ provided by the pencil $s^\times$, as on fig. 3◇15, or use more elegant method based on the Exercise 3.11 below.

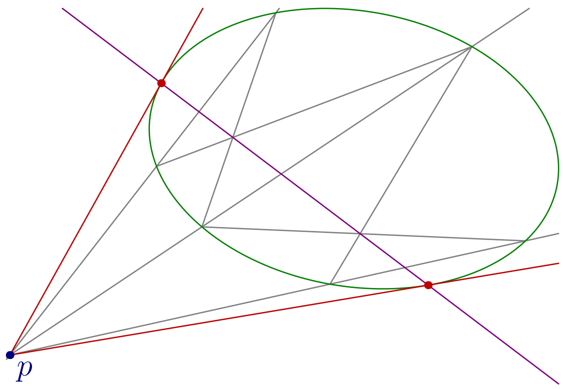

Fig. 3◇15. Drawing the tangent lines.

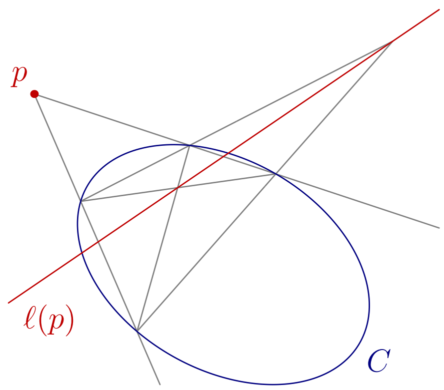

Fig. 3◇16. Drawing the polar.

Exercise 3.11 (Steiner's construction). Shown on fig. 3◇16 is the construction of polar line $\ell(p)$ for a point p with respect to a conic C due to Jacob Steiner[2] (1796 – 1863) and using only the ruler. Explain how and why does it work.

3.3 Pencils of conics. Recall[3] that lines in the space of conics $\mathbb{P}(S^2V^*)$ on the plane $\mathbb{P}_2 = \mathbb{P}(V)$ are called *pencils of conics*. A pencil $L \subset \mathbb{P}(S^2V^*)$ is uniquely described by any pair of distinct conics $C_0 = V(f_0)$, $C_1 = V(f_1)$ from L and consists of the conics $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1)$, where $\lambda = (\lambda_0 : \lambda_1) \in \mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$. The intersection $B = C_0 \cap C_1$ is called *the base set* of the pencil. It does not depend on the choice of basis $C_0, C_1 \in L$, because every conic $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1) \in L$ contains $B = V(f_0) \cap V(f_1)$ for any two distinct conics $C_0 = V(f_0)$, $C_1 = V(f_1)$ in L.

The polynomial $\chi_{(f_0 f_1)}(t_0, t_1) \stackrel{\text{def}}{=} \det(t_0 f_0 + t_1 f_1) \in \Bbbk[t_0, t_1]$ is called *the characteristic polynomial* of the pencil with respect to the base conics C_0, C_1. This is a cubic homogeneous polynomial. Up to multiplication by non zero constants, it does not dependent on a choice of basis in V used for the evaluation of determinant. However, in a contrast with the base set, the characteristic polynomial depends on a choice of basis in the pencil, and a change of basis leads to an invertible linear change of variables (t_0, t_1). Thus, an invariant of the pencil is not the characteristic polynomial

[1] See the Theorem 3.1 on p. 30.

[2] See *J. Steiner*. «Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises: als Lehrgegenstand auf höheren Unterrichts-Anstalten und zur praktischen Benutzung», Ostwald's Klassiker der exakten Wissenschaften, vol. 60.

[3] See n° 1.3.2 on p. 9.

itself but the combinatorial structure of its zero set in $\mathbb{P}_1$. Over algebraically closed field, the latter is either the whole $\mathbb{P}_1$, or one point of multiplicity 3, or a pair of distinct points of multiplicities 1 and 2, or a triple of distinct points, each of multiplicity 1. In the first case, the pencil is called *degenerated*; in the latter case, it is called *simple*. Thus, a pencil is degenerated if and only it consists of singular conics. A non-degenerated pencil over algebraically closed field can contain 1, 2, or 3 degenerated conics, and $\operatorname{Sing} C_0 \cap \operatorname{Sing} C_1 = \varnothing$ for any two different conics C_0, C_1 in the pencil, because a vector $v \in \ker \widehat{f}_0 \cap \ker \widehat{f}_1$ belongs to $\ker(\lambda_0 \widehat{f}_\lambda + \lambda_1 \widehat{f}_1)$ for all $\lambda \in \mathbb{P}_1$. The base set of a non-degenerated pencil over algebraically closed field can consist of 1, 2, 3, or 4 points.

Lemma 3.2
For every conic $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1)$ in a non-degenerated pencil, $\dim \operatorname{Sing} C_\lambda$ is strictly less than the maximal power of $\det(\lambda, t) = \lambda_0 t_1 - \lambda_1 t_0$ dividing the characteristic polynomial $\chi_{(f_0 f_1)}(t_0, t_1)$ in $\Bbbk[t_0, t_1]$.

Proof. Let D be an arbitrary conic of the pencil, and C a smooth conic. Fix a basis in V such that the Gram matrix of C is the identity matrix E, and write A for the Gram matrix of D. Then the conics in pencil (CD) become the Gram matrices $tE + A$, where $t \in \Bbbk$ is a coordinate on affine line $(CD) \smallsetminus C$. The conic D appears for $t = 0$. We have to show that $\dim \ker A$ can not exceed the maximal power of t dividing $\det(tE + A) = t^3 + t^2 \delta_1(A) + t\delta_2(A) + \delta_3(A)$, where $\delta_k(A)$ is the sum of principal $k \times k$ minors in A. This is obvious, because all minors of order $> 3 - k$ in A vanish as soon $\operatorname{rk} A \leqslant 3 - k$. □

Exercise 3.12. Prove that a non-degenerated pencil of conics contains at most one double line.

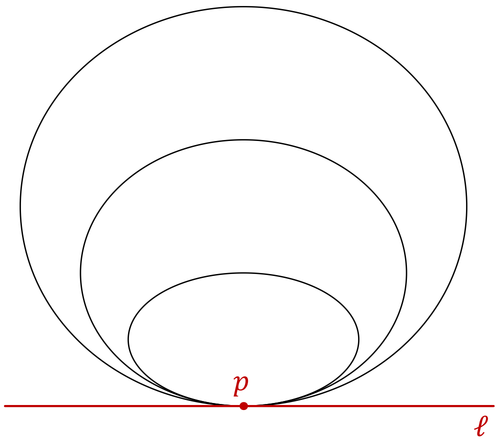

Fig. 3⋄17. A pencil with 1 base point.

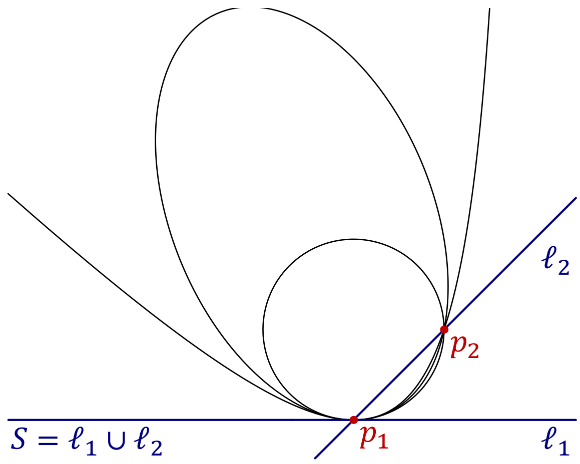

Fig. 3⋄18. A pencil with 2 base points and 1 singular conic.

Example 3.3 (non-degenerated pencil with just one base point)
If the base set of a non-degenerated pencil consists of just one point p, then the only singular conic in the pencil is the double line tangent to any smooth conic of the pencil at the point p. Thus, such a pencil is spanned by a smooth conic $C \ni p$ and the double line $\ell = T_p C$. Note that any two smooth conics in such a pencil have the unique intersection point and share the common tangent line at this point, see fig. 3⋄17.

Example 3.4 (non-degenerated pencils with two base points)
If the base set of a pencil consists of two points $p_1 \neq p_2$, then a singular conic in such pencil has to be either the double line $\ell = (p_1 p_2)$ or a split conic $\ell_1 \cup \ell_2$ such that $p_1 \in \ell_1$, $p_2 \in \ell_2$ and either p_1, p_2 both differ from $\ell_1 \cap \ell_2$, as on fig. 3◇19, or $p_1 = \ell_1 \cap \ell_2$, $p_2 \neq \ell_1 \cap \ell_2$, as on fig. 3◇18.

In the latter case the split conic $\ell_1 \cap \ell_2$ is the only singular conic in the pencil. All the other conics are smooth, touch the line ℓ_1 at p_1, and pass through p_2 like on fig. 3◇18. In particular, any two smooth conics in such a pencil have exactly two different intersection points p_1, p_2 and share the same tangent line at p_1.

The first two possibilities for a singular conic, i.e., the double line $\ell = (p_1 p_2)$ or a split conic $\ell_1 \cup \ell_2$ such that $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2 \in \ell_2 \smallsetminus \ell_2$, can be realized in a pencil with 2 base points only simultaneously.

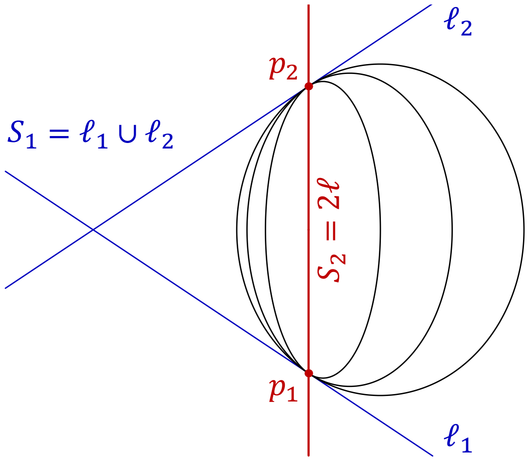

Fig. 3◇19. A pencil with 2 base points and 2 singular conics S_1, S_2.

Exercise 3.13. Prove that all conics in $\mathbb{P}_2$ that touch two given lines ℓ_1, ℓ_2 at two given points $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2 \in \ell_2 \smallsetminus \ell_1$ form a pencil with exactly two singular conics: the double line $\ell = (p_1 p_2)$ and the split conic $\ell_1 \cup \ell_2$.

Both lines ℓ_1, ℓ_2 are uniquely recovered from the double line ℓ and any smooth conic C of the pencil as the tangent lines to C at the intersection points $C \cap \ell$.

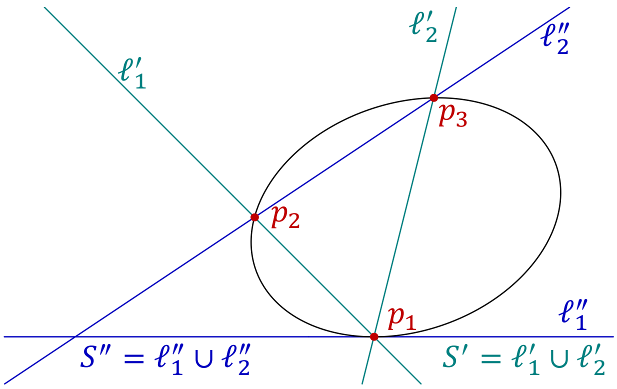

Fig. 3◇20. A pencil with 3 base poins has 2 singular conics.

Example 3.5 (non-degenerated pencil with three base points)
If the base set of a pencil consists of 3 distinct points p_1, p_2, p_3, then these points are not collinear[1]. Hence, such a pencil does not contain a double line. For any split conic $\ell_1 \cup \ell_2$ in the pencil, there are two possibilities: either $p_1 = \ell_1 \cap \ell_2$, $p_2 \in \ell_1 \smallsetminus \ell_2$, $p_3 \in \ell_2 \smallsetminus \ell_1$ or $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2, p_3 \in \ell_2 \smallsetminus \ell_1$. On fig. 3◇20, the first happens for the lines ℓ'_1, ℓ'_2, the second for the lines ℓ''_1, ℓ''_2. If the pencil contains $\ell''_1 \cup \ell''_2$, then every smooth conic from the pencil touches ℓ''_1 at p_1. Note that the split

[1] Otherwise the line passing through them would intersect every smooth conic of the pencil in 3 distinct points.

conic $\ell_1' \cup \ell_2'$ satisfies this property.

EXERCISE 3.14. Prove that all conics passing through 3 given distinct points a, b, c and touching a given line $\ell \ni c$ form a pencil containing exactly 2 singular conics: $(ab) \cup \ell$ and $(ac) \cup (bc)$.

If the pencil contains $\ell_1' \cup \ell_2'$, then all smooth conics in the pencil also have to share the same tangent line at the point p_1, because a line $\ell \ni p_1$ tangent to a smooth conic $C \ni p_1$ touches at p_1 every conic D from the pencil spanned by C and $\ell_1' \cup \ell_2'$. Thus, such a pencil is described by the Exercise 3.14 as well.

EXAMPLE 3.6 (SIMPLE PENCIL OF CONICS)
A pencil of conics over algebraically closed field is simple if and only if it contains three distinct singular conics. Each of these singular conics splits by the Lemma 3.2, and does not pass trough the singular points of two others. Therefore every pair of singular conics has 4 intersection points any 3 of which are non-collinear, see fig. 3⋄21. These 4 points form the base set of pencil.

EXERCISE 3.15. Prove that all conics passing through 4 given points a, b, c, d no 3 of which are collinear form a simple pencil containing exactly 3 singular conics formed by the pairs of opposite sides in quadrangle $abcd$.

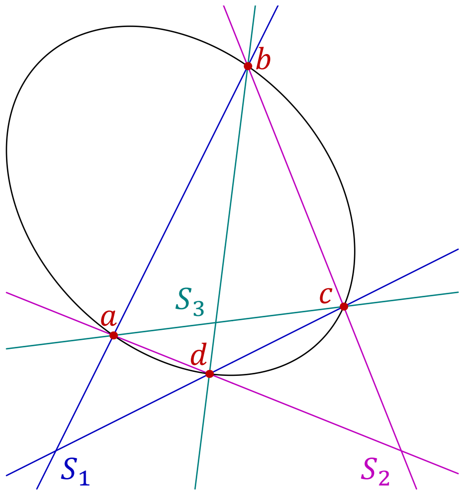

Fig. 3⋄21. 3 singular conics and 4 base points of a simple pencil.

Thus, a simple pencil of conics is uniquely determined by its base points a, b, c, d. In homogeneous coordinates $x = (x_0 : x_1 : x_2)$ on $\mathbb{P}_2$, the equations of conics from this pencil can be written as

$$\frac{\det(x, a, b) \cdot \det(x, c, d)}{\det(x, a, d) \cdot \det(x, b, c)} = \frac{\lambda_0}{\lambda_1},$$

where $\lambda = (\lambda_0 : \lambda_1)$ runs through $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$.

All the previous examples of pencils can be viewed as degenerations of a simple pencil appearing when some of the base points stick together. For $a, b \to p_1$, $c = p_2$, $d = p_3$, we get the pencil on fig. 3⋄20. For $a, b \to p_1$, $c, d \to p_2$, we come to the pencil on на fig. 3⋄19. When $a, b, c \to p_1$, $d = p_2$, we get fig. 3⋄18. Finally, on fig. 3⋄17, all 4 base points are collapsed to one point p.

3.3.1 The hypersurface of singular conics. The singular conics in $\mathbb{P}_2 = \mathbb{P}(V)$ form a cubic hypersurface $S = V(\det)$ in the space $\mathbb{P}_5 = \mathbb{P}(S^2)$ of all conics. The roots of characteristic polynomial $\chi_{(f_0 f_1)}(t_0, t_1)$ correspond to the intersection points of S with the line $L = (C_0 C_1)$ spanned by conics $C_0 = V(f_0)$, $C_1 = V(f_1)$. The character of intersection $S \cap L$ completely determines the geometric properties of the pencil L. A simple pencil L intersects S in 3 distinct points with the multiplicity 1 at each point. If L touches S at a smooth point of S and intersects S with the multiplicity 1 in one more point, then the pencil L looks as on fig. 3⋄20, where the split conic with singularity at a base point of L corresponds to the touch point of L with S. If L passes through a singular point of S and intersects S once more in another point, then L looks as on fig. 3⋄19, where the double line corresponds to the singular intersection point of L and S. If L intersects S with the multiplicity 3 in one smooth point of S, the pencil looks as on fig. 3⋄18. The most degenerated pencil shown on fig. 3⋄17 is provided by a line L intersecting S with the multiplicity 3 in one singular point of S.

Comments to some exercises

EXRC. 3.1. This is a particular case of the Exercise 1.12.

EXRC. 3.2. Draw the cross-axix ℓ by joining $(a_1b_2) \cap (b_1, a_2)$ and $(c_1b_2) \cap (b_1, c_2)$). Then draw a line through b_1 and $\ell \cap (x, b_2)$. This line crosses ℓ_2 in $\varphi(x)$.

EXRC. 3.3. Let two tangent lines to C drown from x be given by linear equations $\xi(x) = 0$, $\eta(x) = 0$, and let the line ℓ_1 be the second of them. Then $\xi, \eta \in \mathbb{P}_2^\times$ are the intersection points of the dual conic $C^\times \subset \mathbb{P}_2$ and the line $\operatorname{Ann} x \subset \mathbb{P}_2^\times$. To find them, we need to solve a quadratic equation whose coefficients are polynomials in the coordinates of the point x and the elements of the Gram matrix of conic C. One root of this equation leads to the given point $\eta \in \mathbb{P}_2$ and therefore is known. Then the second root is a rational function of the first root and the coefficients of quadratic equation by the Vieta formula.

EXRC. 3.4. The arguments are dual to those from the Exercise 3.3.

EXRC. 3.6. Let $c_1, c_2 \in C \smallsetminus \{a_1, a_2, a_3, a_4\}$. Parametrize the pencils $c_1^\times$ and $c_2^\times$ by some lines $\ell_1 \not\ni c_1$ and $\ell_2 \not\ni c_2$ respectively, and write a_i', a_i'' for the images of points a_i under the projections $c_i : D \xrightarrow{\sim} \ell_i$. Then $[a_1, a_2, a_3, a_4] = [a_1', a_2', a_3', a_4'] = [a_1'', a_2'', a_3'', a_4'']$, where the second equality holds, because the composition of projections $(c_2 : D \xrightarrow{\sim} \ell_2) \circ (c_1 : \ell_1 \xrightarrow{\sim} D)$ is a homography $\ell_1 \xrightarrow{\sim} \ell_2$ sending $a_i \mapsto a_i''$ for all i (comp. with n° 3.1.3 on p. 28). Since any linear projective automorphism $\varphi : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ induces the homography of the pencils of lines $a^\times \xrightarrow{\sim} \varphi(a)^\times$, the second statement of the problem holds as well.

EXRC. 3.8. This is the smooth conic passing through p, q, a, b, c.

EXRC. 3.10. For given $p, q \in \mathbb{P}_1$, the equality $[p, q, x, y] = -1$ allows to express $x = x_0/x_1$ and $y = y_0/y_1$ through one other rationally. Hence, by the Lemma 3.1 on p. 26, a homography $\mathbb{P}_1 \to \mathbb{P}_1$ is provided by the map sending a point $x \in \mathbb{P}_1$ to the point $y \in \mathbb{P}_1$ such that $[p, q, x, y] = -1$. It is involutive[1], because $[p, q, x, y] = -1 = [p, q, y, x]$. Since it keeps both p, q fixed, it coincides with $\sigma_{p,q}$.

EXRC. 3.13. For a point p and line ℓ in $\mathbb{P}_2 = \mathbb{P}(V)$, the conics $C = V(f) \subset \mathbb{P}_2$ such that ℓ is the polar of p with respect to C form a projective subspace of codimension 2 in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$. Indeed, associated with $p \in V$ is the linear map

$$\operatorname{pl}_p : S^2V^* \to V^*, \quad q \mapsto \hat{q}(p), \tag{3-8}$$

which sends a quadratic form q to the covector $\hat{q}(p) : V \to \mathbb{k}$, and $\dim \ker \operatorname{pl}_v = \dim S^2V^* - \dim V^* = 3$ when $\dim V = 3$. Thus, the preimage of dimension 1 subspace $\operatorname{Ann}(\ell) \in V^*$ under the map (3-8) has dimension 4, that is, codimension 2. Its projectivisation is of codimension 2 as well. In particular, for $p \in \ell$, this gives what we have stated. Futher, two subspaces of codimension 2 in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$ formed, respectively, by conics touching the lines ℓ_1, ℓ_2 at the points $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2 \in \ell_2 \smallsetminus \ell_1$ are intersecting at least along a line. If their intersection would a plane, then for any pair of points $a, b\mathbb{P}_2$ there would be a conic passing through a, b and touching ℓ_1, ℓ_2 at p_1, p_2 respectively. For $a \in \ell \smallsetminus \{p_1, p_2\}$, $b \notin \ell \cup \ell_1 \cup \ell_2$, such the conic must split into the line ℓ and another line different from ℓ, ℓ_1, ℓ_2. Hence, this conic can not intersect ℓ_1, ℓ_2 with multiplicities 2 in p_1, p_2 simultaneously.

[1] Do you see that in the affine chart whose infinity is p, the this homography is nothing but the central symmetry with respect to q?

EXRC. 3.14. The first follows from the fact that $\ell_1'' \cup \ell_2''$ also touches ℓ at p_1. The second is similar to the Exercise 3.13: use the facts that conics passing through a given point form a hyperplane, whereas conics touching a given line at a given point form a subspace of codimension 2 in the space of conics.

EXRC. 3.15. Four hyperplanes in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$ formed by the conics passing through a, b, c, d are linearly independent, because for any 3 of the points, there is a split conic passing through them but not through the remaining fourth point. Hence, these 4 hyperplanes are intersecting along a line. The split conics formed by pairs of opposite sides in quadrangle $abcd$ lie in the pencil. This forces the pencil to be simple.