§4 Tensor Guide

4.1 Tensor products and Segre varieties. Let $V_1, V_2, \ldots, V_n$ and $W$ be vector spaces of dimensions $d_1, d_2, \ldots, d_n$ and $m$ over a field $k$. A map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ is called multilinear, if it is linear in each argument when all the other are fixed:

$$\varphi(\ldots, \lambda v' + \mu v'', \ldots) = \lambda \varphi(\ldots, v', \ldots) + \mu \varphi(\ldots, v'', \ldots).$$

Multilinear maps $V_1 \times V_2 \times \cdots \times V_n \to W$ form a vector space denoted $\text{Hom}(V_1, V_2, \ldots, V_n; W)$. As soon as bases $e_1, e_2, \ldots, e_m \in W$ and $e^{(1)}_1, e^{(1)}_2, \ldots, e^{(1)}_{d_1} \in V_1$, $1 \leq i \leq n$, are fixed, every multilinear map $\varphi \in \text{Hom}(V_1, V_2, \ldots, V_n; W)$ can be uniquely described by the values on all collections of basis vectors:

$$\varphi(e^{(1)}_1, e^{(2)}_1, \ldots, e^{(n)}_1) = \sum_v a_v^{(a_1, a_2, \ldots, a_n)} e_v \in W,$$

that is, by $m \cdot \prod d_i$ constants $a_v^{(a_1, a_2, \ldots, a_n)} \in k$, which can be organized in the matrix of dimension $(n + 1)$ and size $m \times d_1 \times d_2 \times \cdots \times d_n$. The multilinear map $\varphi$ corresponding to such a matrix sends a collection of vectors $v_1, v_2, \ldots, v_n$, where $v_i = \sum_{a_i=1}^{d_i} x^{(i)}_{a_i} e^{(i)}_{a_i} \in V_i$ for $1 \leq i \leq n$, to the vector

$$\varphi(v_1, v_2, \ldots, v_n) = \sum_{v=1}^{m} \left( \sum_{a_1, a_2, \ldots, a_n} a_v^{(a_1, a_2, \ldots, a_n)} \cdot x^{(1)}_{a_1} \cdot x^{(2)}_{a_2} \cdots x^{(n)}_{a_n} \right) e_v \in W.$$

Thus, $\dim \text{Hom}(V_1, V_2, \ldots, V_n; W) = \dim W \cdot \prod_i \dim V_i$.

Exercise 4.1. Check that a) a collection of vectors $v_1, v_2, \ldots, v_n \in V_1 \times V_2 \times \cdots \times V_n$ does not contain the zero vector if and only if there exists a multilinear map $\varphi$ such that $\varphi(v_1, v_2, \ldots, v_n) \neq 0$ b) for a linear $F : U \to W$ and multilinear $\varphi : V_1 \times V_2 \times \cdots \times V_n \to U$, the composition $F \circ \varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ is multilinear.

4.1.1 Tensor product of vector spaces. Given a multilinear map

$$\tau : V_1 \times V_2 \times \cdots \times V_n \to U \quad (4-1)$$

and a vector space $W$, composing $\tau$ with linear maps $F : U \to W$ assigns the map

$$\text{Hom}(U, W) \xrightarrow{F \circ \tau} \text{Hom}(V_1, V_2, \ldots, V_n; W) \quad (4-2)$$

which is obviously linear in $F$.

Definition 4.1

A multilinear map $(4-1)$ is called universal if for any vector space $W$, the linear map $(4-2)$ is an isomorphism. In the expanded form, this means that for every vector space $W$ and multilinear map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$, there exist a unique linear operator $F : U \to W$ such that $\varphi = F \circ \tau$, i.e., two solid multilinear arrows in the diagram

1The usual matrices of dimension 2 and size $d \times m$ describe linear maps $V \to W$. 38
are uniquely completed to a commutative triangle by the dashed linear arrow.

**Lemma 4.1**
For every two universal multilinear maps
\[
\tau_1 : V_1 \times V_2 \times \cdots \times V_n \to U_1, \quad \tau_2 : V_1 \times V_2 \times \cdots \times V_n \to U_2,
\]
there exists a unique linear isomorphism \( i : U_1 \to U_2 \) such that \( \tau_2 = i \tau_1 \).

**Proof.** By the universal properties of \( \tau_1, \tau_2 \), there exists a unique pair of linear maps \( F_{21} : U_1 \to U_2 \) and \( F_{12} : U_2 \to U_1 \) that fit in the commutative diagram

![Diagram](image)

Since the factorizations \( \tau_1 = \varphi \circ \tau_1, \tau_2 = \psi \circ \tau_2 \) are unique and hold for \( \varphi = \text{Id}_{U_1}, \psi = \text{Id}_{U_2} \), we conclude that \( F_{21} F_{12} = \text{Id}_{U_2} \) and \( F_{12} F_{21} = \text{Id}_{U_1} \).

**Lemma 4.2**
Given a basis \( e^{(i)}_1, e^{(i)}_2, \ldots, e^{(i)}_{d_i} \in V_i \) for \( 1 \leq i \leq n \), write \( V_1 \otimes V_2 \otimes \cdots \otimes V_n \) for the vector space with basis formed by \( \prod d_i \) formal expressions
\[
e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\alpha_2} \otimes \cdots \otimes e^{(n)}_{\alpha_n}, \quad 1 \leq \alpha_i \leq d_i.
\] (4.3)

Then the multilinear map \( \tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n \) sending every collection of basis vectors \( (e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\alpha_2} \otimes \cdots \otimes e^{(n)}_{\alpha_n}) \in V_1 \times V_2 \times \cdots \times V_n \) to the expression (4.3) is universal.

**Proof.** For a multilinear \( \varphi : V_1 \times V_2 \times \cdots \times V_n \to W \) and linear \( F : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W \), the identity \( \varphi = F \circ \tau \) means exactly that \( F(e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\alpha_2} \otimes \cdots \otimes e^{(n)}_{\alpha_n}) = \varphi(e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\alpha_2} \otimes \cdots \otimes e^{(n)}_{\alpha_n}) \) for all collections of basis vectors.

**Definition 4.2**
The universal multilinear map (4.1) is denoted by
\[
\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n, \quad (v_1, v_2, \ldots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n \quad (4.4)
\]
and called tensor multiplication. The target space \( V_1 \otimes V_2 \otimes \cdots \otimes V_n \) is called the tensor product of spaces \( V_1, V_2, \ldots, V_n \) and its elements are called tensors.

### 4.1.2 Decomposable tensors and Segre varieties
The image of tensor multiplication (4.4) consists of the tensor products \( v_1 \otimes v_2 \otimes \cdots \otimes v_n \) called tensor monomials or decomposable tensors. They do not form a vector space, because the map (4.4) is not linear but multilinear. However, the linear span of decomposable tensors is the whole space \( V_1 \otimes V_2 \otimes \cdots \otimes V_n \). Over an infinite ground field, a random tensor is most likely an indecomposable linear combination of tensor monomials.
Geometrically, the tensor multiplication assigns a map
\[ s : \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \to \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_n) \] (4-5)
sending a collection of dimension 1 subspaces \( k \cdot v_i \subset V_i \) spanned by non zero vectors \( v_i \in V_i \) to the dimension 1 subspace \( k \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_n \subset V_1 \otimes V_2 \otimes \cdots \otimes V_n. \)

**Exercise 4.2.** Verify that the map (4-5) is a well defined and injective.

The map (4-5) is called the **Segre embedding** and its image, i.e., the projectivization of the set of decomposable tensors, is called the **Segre variety**. Since the decomposable tensors linearly span the whole space, the Segre variety is not contained in a hyperplane. Note that the dimension of Segre variety equals \( \sum m_i \), where \( m_i = d_i - 1 \), and is much smaller then \( \dim \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_n) = \prod (1 + m_i) - 1. \) By the construction, the Segre variety is ruled by \( n \) families of projective subspaces of dimensions \( m_1, m_2, \ldots, m_n \). The simplest example of the Segre variety is provided by the Segre quadric from n’ 2.5.1 on p. 22.

**Example 4.1 (Decomposable Linear Maps)**
For any two vector spaces \( U, W \), the bilinear map \( U^* \times W \to \text{Hom}(U, V) \) is provided by sending \( (\xi, w) \in U^* \times W \) to the linear operator \( U \to W \), \( u \mapsto \langle \xi, u \rangle \cdot w \). By the universal property of tensor multiplication, there exists a unique linear map
\[ U^* \otimes V \to \text{Hom}(U, V) \] (4-6)
sending every decomposable tensor \( \xi \otimes w \) to the same operator. Note that this operator has rank 1, its image is spanned by \( w \in W \), and the kernel is \( \text{Ann}(\xi) \subset U \).

**Exercise 4.3.** Check that A) every linear map \( F : U \to W \) of rank 1 equals \( \xi \otimes w \) for appropriate \( \xi \in U^*, w \in W \) uniquely up to proportionality determined by \( F \) b) the linear map (4-6) is an isomorphism for any vector spaces \( U \) and \( V \) of finite dimensions.

Geometrically, the operators of rank 1 form the Segre variety \( S \subset \mathbb{P}_{mn-1} = \mathbb{P}(\text{Hom}(U, W)) \), which is ruled by two families of projective spaces \( \xi \otimes \mathbb{P}(W), \mathbb{P}(U^*) \otimes w \) and is not contained in a hyperplane. If we fix some bases in \( U, W \), write operators \( U \to W \) by their matrices \( A = (a_{ij}) \) in these bases, and use the matrix elements \( a_{ij} \) as the homogeneous coordinates in \( \mathbb{P}(\text{Hom}(V, W)) \), then the Segre variety is described by the equation \( \text{rk} A = 1 \), which encodes the system of homogeneous quadratic equations
\[ \det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{kj} & a_{jk} \end{pmatrix} = a_{ij}a_{jk} - a_{ik}a_{kj} = 0 \]
for all \( 1 \leq i < j \leq \dim W, 1 \leq j < k \leq \dim U \). The Segre embedding
\[ \mathbb{P}(U^*) \times \mathbb{P}(V) = \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \to \mathbb{P}_{mn-1} = \mathbb{P}(\text{Hom}(U, W)) \]
takes a pair of points \( x = (x_1 : x_2 : \cdots : x_n), y = (y_1 : y_2 : \cdots : y_n) \) to the rank 1 matrix \( A(x, y) = y^t \cdot x \) whose \( a_{ij} = x_j y_i \). For \( \dim U = \dim W = 2 \), we get the Segre quadric in \( \mathbb{P}_4 \) discussed in n’ 2.5.1 on p. 22.

**4.2 Tensor Algebra and Contractions.** Given a vector space \( V \), we write \( V^\otimes n = V \otimes V \otimes \cdots \otimes V \) for the tensor product of \( n \) copies of \( V \) and call it the \( n \)th tensor power of \( V \). We also put \( V^\otimes 0 \cong k, V^\otimes 1 \cong V \). The infinite direct sum \( TV \cong \bigoplus_{n \geq 0} V^\otimes n \) is called the tensor algebra of \( V \). This is
an associative (non-commutative) graded algebra with the multiplication provided by the tensor product of vectors. For every basis \( e_1, e_2, \ldots, e_n \) in \( V \), the tensor monomials
\[
e_{v_1} \otimes e_{v_2} \otimes \cdots \otimes e_{v_m}
\] (4-7)
form a basis of \( TV \) over \( \mathbb{k} \). These monomials are multiplied just by writing them sequentially with the sign \( \otimes \) between then. Linear combinations of monomials are multiplied by the usual distributivity rules. Thus, \( TV \) may be thought of as the algebra of polynomials in \( n \) non-commuting variables \( e_{v_i} \). Another name for \( TV \) is the free associative \( \mathbb{k} \)-algebra with unit spanned by the vector space \( V \). This name emphasizes the following universal property of the \( \mathbb{k} \)-linear map
\[
\iota : V \hookrightarrow TV
\] (4-8)
embedding \( V \) into \( TV \) as the subspace \( V^{\otimes 1} \) of linear homogeneous polynomials.

**EXERCISE 4.4.** Prove that for every associative \( \mathbb{k} \)-algebra \( A \) with unit and \( \mathbb{k} \)-linear map \( f : V \rightarrow A \), there exists a unique homomorphism of associative \( \mathbb{k} \)-algebras \( \alpha : TV \rightarrow A \) such that\(^1\) \( f = \alpha \circ \iota \).

Convince yourself that this property characterizes the inclusion (4-8) uniquely up to a unique isomorphism of the target space commuting with the inclusion.

### 4.2.1 Total contraction and duality

There is the canonical pairing between \((V^*)^{\otimes n}\) and \(V^{\otimes n}\) provided by the total contraction, which sends \( \xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \), \( v = v_1 \otimes v_2 \otimes \cdots \otimes v_n \) to
\[
\langle \xi, v \rangle \triangleq \prod_{i=1}^{n} (\xi_i, v_i).
\] (4-9)

Since the right hand side is multilinear in \( v_i \)'s, every collection of \( \xi_i \)'s assigns the well defined linear map \( V^{\otimes n} \rightarrow \mathbb{k} \), which depends on \( \xi_i \)'s also multilinearly. Hence, the contraction of decomposable tensors (4-9) is uniquely extended to the bilinear pairing \( V^* \otimes V^* \rightarrow \mathbb{k} \). For a pair of dual bases \( e_1, e_2, \ldots, e_n \in V \), \( x_1, x_2, \ldots, x_n \in V^* \), the tensor monomials \( e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \) and \( x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_n} \) form the dual bases of \( TV \) and \( TV^* \) with respect to this pairing. In particular, for a finite dimensional vector space \( V \), we have the canonical isomorphism
\[
(V^{\otimes n})^* \simeq (V^*)^{\otimes n}.
\] (4-10)

It follows from the universal property of \( V^{\otimes n} \) that the space \((V^{\otimes n})^*\) of the linear maps \( V^{\otimes n} \rightarrow \mathbb{k} \) is canonically isomorphic to the space of multilinear maps \( V \times V \times \cdots \times V \rightarrow \mathbb{k} \), i.e.,
\[
(V^{\otimes n})^* \simeq \text{Hom}(V, \ldots, V; \mathbb{k}).
\] (4-11)

Combining (4-10) and (4-11) leads to the canonical isomorphism
\[
(V^*)^{\otimes n} \simeq \text{Hom}(V, \ldots, V; \mathbb{k}).
\] (4-12)

It sends a decomposable tensor \( \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \) to the multilinear map \( V \times V \times \cdots \times V \rightarrow \mathbb{k} \) taking \( (v_1, v_2, \ldots, v_n) \mapsto \prod_{i=1}^{n} \xi_i(v_i) \).

\(^1\)In other words, for every \( \mathbb{k} \)-algebra \( A \), the homomorphisms of \( \mathbb{k} \)-algebras \( TV \rightarrow A \) stay in bijection with the \( \mathbb{k} \)-linear maps \( V \rightarrow A \).
4.2.2 Partial contractions. Consider two inclusions\(^1\) of sets

\[
\{1, 2, \ldots, p\} \overset{\iota}{\hookrightarrow} \{1, 2, \ldots, m\} \overset{\jmath}{\hookrightarrow} \{1, 2, \ldots, q\},
\]

and write \(i_v, j_v\) for \(I(v), J(v)\) respectively. Thus, we have two numbered collections of indexes \(I = (i_1, i_2, \ldots, i_m), J = (j_1, j_2, \ldots, j_m)\) staying in the fixed bijection. A partial contraction of \(V^* \otimes p\) and \(V \otimes q\) in indexes \(I, J\) is the linear map

\[
c_j^I : V^* \otimes p \otimes V \otimes q \rightarrow V^* \otimes (p-m) \otimes V \otimes (q-m)
\]

which contracts \(i_v\) th factor of \(V^* \otimes p\) with \(j_v\) th factor of \(V \otimes q\) for every \(v = 1, 2, \ldots, m\) and keeps all the other factors in their initial order:

\[
\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_q \mapsto \prod_{v=1}^m (\langle \xi_{i_v}, v_{j_v} \rangle \cdot (\otimes \xi_i) \otimes (\otimes v_j)). \tag{4-13}
\]

Note that different choices of the maps \(I, J\) lead to the different contraction maps even if the images of \(I, J\) remain unchanged.

**Example 4.2 (inner product between vectors and multilinear forms)**

Let us treat a \(n\)-linear form \(\varphi(v_1, v_2, \ldots, v_n)\) as a tensor from \(V^* \otimes n\) via isomorphism \((4-12)\). The contraction of this tensor with a vector \(v \in V\) in the first tensor factor is a tensor from \(V^* \otimes (n-1)\), which can be considered as an \((n - 1)\)-linear form on \(V\). This form is called the inner product of \(v\) and \(\varphi\) and denoted by \(i_v \varphi\) or \(v \varphi\).

**Exercise 4.5.** Check that \(i_v \varphi(w_1, w_2, \ldots, w_{n-1}) = \varphi(v, w_1, w_2, \ldots, w_{n-1})\).

4.2.3 The linear support of a tensor. Given a tensor \(t \in V^* \otimes n\), the intersection of all vector subspaces \(W \subset V\) such that \(t \in W^* \otimes n\) is called the linear support of \(t\) and denoted by \(\text{Supp}(t) \subset V\).

It follows from the next the **Exercise 4.6** that \(\text{Supp}(t)\) is the unique minimal\(^2\) subspace in \(V\) among those \(W \subset V\) for which \(t \in W^* \otimes n\).

**Exercise 4.6.** For any subspaces \(U, W \subset V\), verify that \(U^* \otimes n \cap W^* \otimes n = (U \cap W)^* \otimes n\) in \(V^* \otimes n\).

The dimension of \(\text{Supp}(t)\) is called the rank of \(t\) and denoted by \(\text{rk} t \overset{\text{def}}{=} \dim \text{Supp}(t)\). We say that \(t\) is degenerated if \(\text{rk} t < \dim V\). In this case, the number of variables in the expansion of \(t\) through the basis tensor monomials can be reduced by a linear change of variables.

**Exercise 4.7.** Show that if \(\dim \text{Supp}(t) = 1\) and the ground field is algebraically closed, then \(t = \lambda \cdot v^* \otimes n\) for some \(\lambda \in k, v \in V\).

The space \(\text{Supp}(t)\) admits an effective description as a linear span of some finite collection of vectors constructed by means of contraction maps. Namely, for every injective\(^3\) map

\[
J : \{1, 2, \ldots, (n-1)\} \hookrightarrow \{1, 2, \ldots, n\}, \tag{4-14}
\]

write \(\{j_1, j_2, \ldots, j_{n-1}\} \subset \{1, 2, \ldots, n\}\) for the image of \(J\) and \(\hat{J}\) for the remaining index outside \(\text{im} J\). Consider the contraction map

\[
c_{J}^{I} : V^* \otimes (n-1) \rightarrow V, \quad \xi \mapsto c_{(j_1, j_2, \ldots, j_{n-1})}^{(1, 2, \ldots, (n-1))}(\xi \otimes t) \tag{4-15}
\]

\(^{1}\)Not necessary monotonous.

\(^{2}\)With respect to inclusions.

\(^{3}\)Not necessary monotonous.
which couples $\nu$ th tensor factor of $V^{* \otimes (n-1)}$ with $j_{\nu}$ th tensor factor of $t$ for all $1 \leq \nu \leq (n-1)$. The result of such contraction is obviously a linear combination of $j$ th tensor factors of $t$. Thus, it belongs to $\text{Supp}(t)$.

**THEOREM 4.1**
For every $t \in V^{\otimes n}$, the linear support $\text{Supp}(t) \subseteq V$ is spanned by the images of all contraction maps (4.15) coming from $n!$ different choices of the map (4.14).

**PROOF.** Let $\text{Supp}(t) = W \subseteq V$. It is enough to check that every linear form $\xi \in V^*$ annihilating all the subspaces $\text{im}(c_1^t)$ annihilates $W$ as well. Assume the contrary: let a linear form $\xi \in V^*$ annihilate all $c_1^t(V^{* \otimes (n-1)})$ but have a non-zero restriction on $W$. Chose a basis $\xi_1, \xi_2, \ldots, \xi_d \in V^*$ such that $\xi_1 = \xi$ and the restrictions of $\xi_1, \xi_2, \ldots, \xi_k$ on $W$ form a basis in $W^*$. Expand $t$ through the tensor monomials built from the dual basis vectors $w_1, w_2, \ldots, w_k \in W$. The value

$$
\xi \left( c_1^t(\xi_{v_1} \otimes \xi_{v_2} \otimes \cdots \otimes \xi_{v_{n-1}}) \right)
$$

is equal to the complete contraction of $t$ with the basic monomial $\xi_1 \otimes \xi_{v_1} \otimes \xi_{v_2} \otimes \cdots \otimes \xi_{v_{n-1}}$ in the order of coupling prescribed by $J$. This contraction kills all tensor monomials in the expansion of $t$ except for the one, dual to the monomial obtained from $\xi_1 \otimes \xi_{v_1} \otimes \xi_{v_2} \otimes \cdots \otimes \xi_{v_{n-1}}$ by some permutation of factors depending on $J$. Thus, the result of contraction is equal to the coefficient of some monomial containing $w_1$ in the expansion of $t$. Since every such monomial can be reached by appropriate choice of $J$, we conclude that $w_1 \notin \text{Supp}(t)$. Contradiction.

4.3 **Symmetric and grassmannian algebras.** A multilinear map $\varphi : V \times V \times \cdots \times V \rightarrow U$ is called *symmetric* if it remains unchanged under permutations of the arguments, and *alternating* if it vanishes as soon some of the arguments coincide.

**EXERCISE 4.8.** Verify that under a permutation of the arguments, the value of an alternating multilinear map is multiplied by the sign of permutation. Convince yourself that this property implies the alternating property if $\text{char} \, k \neq 2$.

We write $\text{Sym}^n(V, U) \subseteq \text{Hom}(V, \ldots, V; U)$ and $\text{Alt}^n(V, U) \subseteq \text{Hom}(V, \ldots, V; U)$ for subspaces of symmetric and alternating multilinear maps. Everything said about the universal multilinear maps in 4.1.1 on p. 38 makes sense separately for the symmetric and alternating maps as well. The universal symmetric multilinear map is denoted by

$$
\sigma : V \times V \times \cdots \times V \rightarrow S^nV, \quad (v_1, v_2, \ldots, v_n) \mapsto v_1 v_2 \cdots v_n, \tag{4-16}
$$

and called the *commutative* multiplication of vectors. Its target space $S^nV$ is called the $n$ th *symmetric power* of $V$. The universal alternating multilinear map is denoted by

$$
\alpha : V \times V \times \cdots \times V \rightarrow A^nV, \quad (v_1, v_2, \ldots, v_n) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_n, \tag{4-17}
$$

and called the *exterior* \footnote{Also known as *grassmannian* or *super-commutative.*} multiplication of vectors. Its target space $A^nV$ is called the $n$ th *exterior power* of $V$. The universal symmetric and alternating multilinear maps are unique up to a unique isomorphism of the target space commuting with the universal map. The both can be constructed for all $n$ at once by factorizing the tensor algebra $\mathcal{T}V$ by appropriate two-sided ideals.
4.3.1 The symmetric algebra. Write $I_{\text{com}} \subset TV$ for a two-sided ideal spanned by all the differences
\[ u \otimes w - w \otimes u, \quad u, w \in V. \tag{4-18} \]
This ideal is obviously homogeneous in the sense that $I_{\text{com}} = \bigoplus_{n \geq 0} (I_{\text{com}} \cap V^\otimes n)$, and the degree $n$ component $I_{\text{com}} \cap V^\otimes n$ of $I_{\text{com}}$ is linearly generated over $\mathbb{k}$ by all differences of the form
\[ (\cdots \otimes v \otimes w \otimes \cdots) - (\cdots \otimes w \otimes v \otimes \cdots), \tag{4-19} \]
where the both terms are decomposable of degree $n$ and vary only in the order of $v, w$. The factor algebra $SV \cong TV/I_{\text{com}}$ is called the symmetric algebra of $V$. The multiplication in $SV$ comes from the tensor multiplication in $TV$ and is commutative, because of the relations $uw = wu$ appearing after the factorization through (4-18). The symmetric algebra is graded
\[ SV = \bigoplus_{n \geq 0} S^n V, \quad \text{where } S^n V \cong V^\otimes n / (I_{\text{com}} \cap V^\otimes n). \]

**Exercise 4.9.** Show that for every basis $e_1, e_2, \ldots, e_d \subset V$, the monomials $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ form a basis of $SV$ over $\mathbb{k}$.

Thus, we get an isomorphism of algebras $SV \cong \mathbb{k}[e_1, e_2, \ldots, e_d]$. Under this isomorphism, $S^n V$ turns to the subspace of homogeneous polynomials of degree $n$.

**Exercise 4.10.** Deduce from the universal property of tensor multiplication that the map
\[ V \times V \times \cdots \times V \to S^n V \]
provided by the multiplication in $SV$ is the universal symmetric multilinear map. Convince yourself that $SV$ is the free commutative $\mathbb{k}$-algebra spanned by $V$ in the sense that for every commutative $\mathbb{k}$-algebra $A$ and $\mathbb{k}$-linear map $f : V \to A$, there exists a unique homomorphism of $\mathbb{k}$-algebras $\overline{f} : SV \to A$ such that $f = \overline{f} \circ \iota$, where $\iota : V \hookrightarrow SV$ embeds $V$ in $SV$ as the space of linear homogeneous polynomials. Show that the latter embedding is uniquely characterized by the previous universal property up to a unique isomorphism commuting with $\iota$.

4.3.2 The exterior algebraootnote{Also known as the grassmannian algebra or free super-commutative algebra of $V$.} of a vector space $V$ is defined as the factor algebra $AV \cong TV/I_{\text{alt}}$, where $I_{\text{alt}} \subset TV$ is the two-sided ideal generated by all tensor squares $v \otimes v, v \in V$.

**Exercise 4.11.** Check that the space $I_{\text{alt}} \cap V^\otimes 2$ contains all sums $v \otimes w + w \otimes v, v, w \in V$, and is linearly generated over $\mathbb{k}$ by these sums if char $\mathbb{k} \neq 2$.

The ideal $I_{\text{alt}}$ also splits in the direct sum of homogeneous components
\[ I_{\text{alt}} = \bigoplus_{n \geq 0} (I_{\text{alt}} \cap V^\otimes n). \]
The degree $n$ component $I_{\text{alt}} \cap V^\otimes n$ is spanned by decomposable tensors of the form
\[ (\cdots \otimes v \otimes w \otimes \cdots), \quad v \in V. \]
By the Exercise 4.11, all the sums $(\cdots \otimes v \otimes w \otimes \cdots) + (\cdots \otimes w \otimes v \otimes \cdots)$ belong to $I_{\text{alt}} \cap V^\otimes n$ as well and linearly generate it over $\mathbb{k}$ as soon char $\mathbb{k} \neq 2$. The multiplication in $AV$ is called the
**4.3. Symmetric and exterior algebras**

**exterior**\(^1\) multiplication and denoted by the wedge sign \(\wedge\). Note that for any \(u, w \in V\), the relations \(u \wedge u = 0\) and \(u \wedge w = -w \wedge u\) hold in \(A^2V\). Hence, under a permutation of factors, the exterior product of vectors is multiplied by the sign of permutation:

\[
\forall g \in S_k \quad v_1 \wedge v_2 \wedge \cdots \wedge v_k = \text{sgn}(g) \cdot v_{g_1} \wedge v_{g_2} \wedge \cdots \wedge v_{g_k}.
\]

This property of a multiplication is known as the **super-commutativity**. Like the symmetric algebra, the exterior algebra is graded:

\[
A^\ast V = \bigoplus_{n \geq 0} A^n V, \quad \text{where } A^n V \triangleq V^\otimes n / (I_{\text{alt}} \cap V^\otimes n).
\]

**EXERCISE 4.12.** Deduce from the universal property of tensor multiplication that the map

\[
V \times V \times \cdots \times V \to A^n V
\]

provided by the exterior multiplication in \(AV\) is the universal alternating multilinear map. Convince yourself that \(AV\) is the free super-commutative \(k\)-algebra spanned by \(V\) in the sense that for every super-commutative \(k\)-algebra \(A\) and \(k\)-linear map \(f : V \to A\), there exists a unique homomorphism of \(k\)-algebras \(\bar{f} : SV \to A\) such that \(f = \bar{f} \circ \iota\), where \(\iota : V \hookrightarrow SV\) embeds \(V\) in \(AV\) as the subspace \(A^2 V = V^\otimes k\). Show that the latter embedding is uniquely characterized by the previous universal property up to a unique isomorphism commuting with \(\iota\).

**PROPOSITION 4.1**

For every basis \(e_1, e_2, \ldots, e_d\) in \(V\) the grassmannian monomials \(e_I \triangleq e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}\), numbered by strictly increasing multi-indices \(I = (i_1, i_2, \ldots, i_n), 1 \leq i_1 < i_2 < \cdots < i_n \leq d\), form a basis of \(A^n V\).

**PROOF.** Write \(U\) for the vector space of dimension \(\binom{d}{n}\) with the basis formed by symbols \(\xi_I\), where \(I = (i_1, i_2, \ldots, i_n)\) runs through all strictly increasing sequences of length \(n\) in \(1, 2, \ldots, d\). Consider the multilinear map \(\alpha : V \times V \times \cdots \times V \to U\) that takes an arbitrary collection \(e_{j_1}, e_{j_2}, \ldots, e_{j_n}\) of the basis vectors from \(V\) to \(\alpha(e_{j_1}, e_{j_2}, \ldots, e_{j_n}) = \text{sgn}(\sigma) \cdot \xi_I\), where \(I = (j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(n)})\) is the strictly increasing permutation of the indexes \(j_1, j_2, \ldots, j_n\) and we put \(\alpha(e_{j_1}, e_{j_2}, \ldots, e_{j_n}) = 0\) when some of \(j_{\sigma(\cdot)}\)'s coincide. For any alternating multilinear map \(\varphi : V \times V \times \cdots \times V \to W\), there exists a unique linear operator \(F : U \to W\) such that \(\varphi = F \circ \alpha\): the action \(F\) on the basis of \(U\) has to be \(F(\xi_{(i_1, i_2, \ldots, i_n)}) = \varphi(e_{i_1}, e_{i_2}, \ldots, e_{i_n})\). Thus, \(\alpha\) is the universal alternating multilinear map. Hence, there exists an isomorphism \(U \cong A^n V\) sending \(\xi_I \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I\). \(\square\)

**COROLLARY 4.1**

\[
\dim A^n V = \binom{d}{n}, \quad \text{where } d = \dim V. \quad \text{In particular, } A^d V = 0 \text{ for } n > d, \quad \text{and } \dim AV = 2^d.
\]

**EXERCISE 4.13.** Check that \(\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha\) for any \(\alpha \in A^a V, \beta \in A^b V\), and describe the centre\(^2\) \(Z (AV)\).

\(^1\)Or grassmannian, or super-commutative

\(^2\)That is, all elements commuting with every element of the algebra.
4.3.3 Grassmannian polynomials. It follows from the Proposition 4.1 that every choice of basis \( e_1, e_2, \ldots, e_d \) in a vector space \( V \) assigns the isomorphism of \( k \)-algebras

\[
\Lambda V \xrightarrow{} k \langle e_1, e_2, \ldots, e_d \rangle,
\]

where \( k \langle e_1, e_2, \ldots, e_d \rangle \) stays for the algebra of **grassmannian polynomials**, i.e., polynomials with coefficients from \( k \) in the variables \( e_i \) satisfying the relations \( e_i \wedge e_i = 0 \) and \( e_i \wedge e_j = -e_j \wedge e_i \). When we work with the grassmannian polynomials, we always write \( I = (i_1, i_2, \ldots, i_n) \) for a strictly increasing collection of indexes, \( I = (i_1, \ldots, i_n) = \{1, 2, \ldots, d\} \setminus I \) for the complementary strictly increasing collection, and \( \#I \equiv n \) for the **length** of \( I \). The sum \( \#I \equiv \sum \nu \nu \nu \) is called the **weight** of \( I \).

**Exercise 4.14.** Check that \( e_i \wedge e_i = (-1)^{|I|+\frac{1}{2}\#(1+|I|)} \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_d \).

**Example 4.3 (linear substitution of variables)**

Let the variables \( e_1, e_2, \ldots, e_n \) be linearly expressed through the variables \( \xi_1, \xi_2, \ldots, \xi_m \) as

\[
e_i = \sum_j a_{ij} \xi_j
\]

for some \( n \times m \) matrix \( A = (a_{ij}) \). Then the grassmannian monomials \( e_i \) are expressed through \( \xi_I \) as

\[
e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = \left( \sum_j a_{i_1j} \xi_{j_1} \right) \wedge \left( \sum_j a_{i_2j} \xi_{j_2} \right) \wedge \cdots \wedge \left( \sum_j a_{i_nj} \xi_{j_n} \right) = \\
= \sum_{1 \leq j_1 < j_2 < \cdots < j_n} \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{i_1j_1} a_{i_2j_2} \cdots a_{i_nj_n} \xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n}
\]

where \( J \) runs through increasing collections of length \( n \) and \( a_{ij} \) denotes the \( n \times n \) minor of \( A \) situated in the rows \( i_1, i_2, \ldots, i_n \) and columns \( j_1, j_2, \ldots, j_n \).

**Example 4.4 (multirow cofactor expansions of determinant)**

Let us perform the substitution (4.20) in the identity from the Exercise 4.14 using a square \( d \times d \) matrix \( A \). The left hand side of the identity turns to

\[
\left( \sum_k a_{IK} \xi_K \right) \wedge \left( \sum_L a_{IL} \xi_L \right) = (-1)^{\frac{1}{2}\#(1+|I|)} \sum_k (-1)^{|K|} a_{IK} a_{JK} \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d.
\]

The right hand side becomes \((-1)^{\frac{1}{2}\#(1+|I|)} (-1)^{|I|} \det(A_{ij}) \cdot \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d \). Thus, for every collection \( I = (i_1, i_2, \ldots, i_n) \) of rows in a square matrix \( A = (a_{ij}) \), the following relation holds

\[
\sum_k (-1)^{|K|+|I|} a_{IK} a_{JK} = \det(A_{ij})
\]

where the summation goes over all \( n \times n \) minors \( a_{IK} \) situated in the rows \((i_1, i_2, \ldots, i_n)\).

If we replace \( I \) by another collection \( J \) complementary to the other \( J \neq I \), then we get in the right hand side \( e_I \wedge e_J = 0 \). Thus, for every \( J \neq I \),

\[
\sum_k (-1)^{|K|+|J|} a_{IK} a_{JK} = 0.
\]
The identities (4.21) and (4.22) are known as the Laplace relations. They generalize the cofactor expansions of determinants. If we organize $n \times n$ minors of $A$ and their complements in two $(d_1 \times d_1) \times (d_2 \times d_2)$ matrices $A_n = (a_{ij})$ and $A_n'$ = $(a_{ij}')$, where $^1 a_{ij} = (-1)^{|i|+|j|} a_{ij}$, then all the Laplace relations can be combined in the one matrix identity $A_n \cdot A_n' = \det A \cdot E$.

**Exercise 4.15.** Write the Laplace relations for multicolour cofactor expansions and prove that $A_n \cdot A_n' = \det A \cdot E$ as well.

**Example 4.5 (Reduction of Grassmannian Quadratic Form)**
Certainly, a grassmannian quadratic form can not be reduced to a «sum of squares» like in the Proposition 2.1 on p. 16. However, every homogeneous grassmannian polynomial of degree two over an arbitrary field $k$ takes in appropriate coordinates the form

$$\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots + \xi_{2r-1} \wedge \xi_{2r},$$

(4.23)
called the Darboux normal form. To achieve it for a given $\omega \in A^2V$, we renumber the initial basis $e_1, e_2, \ldots, e_n$ of $V$ in such a way that $\omega = e_1 \wedge (\alpha_2 e_2 + \cdots + \alpha_n e_n) + e_2 \wedge (\beta_3 e_3 + \cdots + \beta_n e_n) + (\text{terms without } e_1, e_2)$, where $\alpha_2 \neq 0$. Then we pass to the new basis $\{e_1, \xi_2, \ldots, e_n\}$ which has $\xi_2 = \alpha_2 e_2 + \cdots + \alpha_n e_n$. The substitution $e_2 = (\xi_2 - \beta_3 e_3 - \cdots - \beta_n e_n) / \alpha_2$ in $\omega$ leads to

$$\omega = e_1 \wedge \xi_2 + (\gamma_2 e_2 + \cdots + \gamma_n e_n) + (\text{terms without } \xi_2) = (e_1 - \gamma_3 e_3 - \cdots - \gamma_n e_n) \wedge \xi_2 + (\text{terms without } e_1, \xi_2).$$

Now we pass to the basis $\{\xi_1, \xi_2, e_3, \ldots, e_n\}$, where $\xi_1 = e_1 - \gamma_3 e_3 - \cdots - \gamma_n e_n$. In this basis,

$$\omega = \xi_1 \wedge \xi_2 + (\text{terms without } \xi_1, \xi_2)$$

and we can continue by induction.

** Convention 4.1.** In the rest of §4 we assume on default that char$(k) = 0$.

### 4.4 Symmetric and alternating tensors

The symmetric group $S_n$ acts on $V^\otimes n$ by permutations of factors in decomposable tensors: for $g \in S_n$, we put

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{g(1)} \otimes v_{g(2)} \otimes \cdots \otimes v_{g(n)}.$$  (4.24)

Since the right hand side is multilinear in $v_1, v_2, \ldots, v_n$, this formula assigns the well defined linear map $g : V^\otimes n \to V^\otimes n$.

**Definition 4.3**

A tensor $t \in V^\otimes n$ is called symmetric, if $g(t) = t$ for all $g \in S_n$. A tensor $t \in V^\otimes n$ is called alternating, if $g(t) = \text{sgn}(g) \cdot t$ for all $g \in S_n$. We write $\text{Sym}^n V = \{ t \in V^\otimes n \mid \forall g \in S_n \, \sigma(t) = t \}$ and $\text{Alt}^n V = \{ t \in V^\otimes n \mid \forall g \in S_n \, g(t) = \text{sgn}(g) \}$ for the space of symmetric and alternating tensors respectively. Note that both are the subspaces in $V^\otimes n$, and they should not be confused with the quotient spaces $S^n V, A^n V$ of $V^\otimes n$.

1 Note that $I, J$ swap places.
4.4.1 Standard bases. For every basis $e_1, e_2, \ldots, e_d$ in $V$, a basis of $\text{Sym}^n V$ is formed by the complete symmetric tensors

$$e_{[m_1, m_2, \ldots, m_d]} \overset{\text{def}}{=} \left( \text{the sum of all tensor monomials containing } m_1 \text{ factors } e_1, m_2 \text{ factors } e_2, \ldots, m_d \text{ factors } e_d \right)$$

(4-25)

because all the summands appear in the expansion of every symmetric tensor $t$ with equal coefficients. The tensors (4-25) are indexed by the collections of non-negative integers $(m_1, m_2, \ldots, m_d)$ such that $\sum m_i = n$.

Exercise 4.16. Make it sure that the sum (4-25) consists of $\frac{n!}{m_1!m_2!\cdots m_d!}$ terms.

Similarly, a basis of $\text{Alt}^n V$ is formed by the complete alternating tensors

$$e_{\langle i_1, i_2, \ldots, i_n \rangle} \overset{\text{def}}{=} \sum_{g \in S_n} \text{sgn}(g) \cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}}$$

(4-26)

numbered by increasing sequences $1 \leq i_1 < i_2 < \cdots < i_n \leq d$.

4.5 Polarization of commutative polynomials. The quotient map $V^{\otimes n} \rightarrow S^n V$ sends every summand of (4-25) to the same commutative monomial $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. Thus, this map sends $e_{[m_1, m_2, \ldots, m_d]}$ to $\frac{m_1!m_2!\cdots m_d!}{n!} \cdot x_{[m_1, m_2, \ldots, m_d]} \in \text{Sym}^n V^*$, which can be viewed as the symmetric multilinear map $f : V \times V \times \cdots \times V \rightarrow \mathbb{k}$ acting on a collection of vectors $v_1, v_2, \ldots, v_n \in V \times V \times \cdots \times V$ via the complete contraction with $v_1 \otimes v_2 \otimes \cdots \otimes v_n$.

Exercise 4.17. Verify that for every $v \in V$, the complete contraction of $v^{\otimes n}$ with

$$\frac{m_1!m_2!\cdots m_d!}{n!} \cdot x_{[m_1, m_2, \ldots, m_d]}$$

is equal to the result of evaluation of monomial $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \in \mathbb{k}[x_1, x_2, \ldots, x_n]$ on the coordinates of $v$.

We conclude that the polynomial function $f : \mathbb{k}(V) \rightarrow \mathbb{k}$ attached to a homogeneous polynomial $f \in S^n V$ in n' 1.1.2 on p. 3 is described in coordinate-free terms as $f(v) = \tilde{f}(v, v, \ldots, v)$, where $\tilde{f} \in \text{Sym}^n V^* \subset V^{\otimes n}$ is the unique symmetric tensor mapped to $f$ under factorization through the commutativity relations and considered as a symmetric multilinear map $V \times V \times \cdots \times V \rightarrow \mathbb{k}$. For $n = 2$, we get the polarization of quadratic forms considered in n' 2.1.1 on p. 16.

Since the value $\tilde{f}(v_1, v_2, \ldots, v_n)$ does not depend on the order of arguments, we write

$$\tilde{f} \left( w_1^{k_1}, w_2^{k_2}, \ldots, w_s^{k_s} \right)$$
when the collection \((v_1, v_2, \ldots, v_n)\) consists of \(k_1\) vectors \(w_1\), \(k_2\) vectors \(w_2\), \ldots, \(k_s\) vectors \(w_s\).

**Exercise 4.18.** For any polynomial \(f \in S^m V^*\) and vectors \(v_1, v_2, \ldots, v_k \in V\), show that
\[
\begin{align*}
f \left( v_1 + v_2 + \cdots + v_k \right) &= \tilde{f} \left( (v_1 + v_2 + \cdots + v_k)^n \right) = \sum_{m_1, m_2, \ldots, m_k} \frac{n!}{m_1! m_2! \cdots m_k!} \tilde{f} \left( v_1^{m_1}, v_2^{m_2}, \ldots, v_k^{m_k} \right), \tag{4-27}
\end{align*}
\]

where the summation goes over all integer \(m_1, m_2, \ldots, m_k\) such that \(m_1 + m_2 + \cdots + m_k = n\) and \(0 \leq m_v \leq n\) for all \(v\).

**Proposition 4.2**
The complete polarization of a homogeneous polynomial \(f \in S^m V^*\) on a vector space\(^1\) \(V\) over a field of zero characteristic can be computed by the formula
\[
n! \cdot \tilde{f}(v_1, v_2, \ldots, v_n) = \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} f \left( \sum_{i \in I} v_i \right), \tag{4-28}
\]
where the left summation goes over all proper subsets \(I \subseteq \{1, 2, \ldots, n\}\), including \(I = \emptyset\), for which we put \(\#\emptyset = 0\).

**Example 4.6**
For homogeneous quadratic and cubic polynomials \(q \in S^2 V^*, f \in S^3 V^*,\) we get
\[
\begin{align*}
2 \tilde{q}(u, w) &= q(u + w) - q(u) - q(w), \\
6 \tilde{f}(u, v, w) &= f(u + v + w) - f(u + v) - f(u + w) - f(v + w) + f(u) + f(v) + f(w).
\end{align*}
\]

**Proof of the Proposition 4.2.** In the expansion (4-27) for
\[
f(v_1 + v_2 + \cdots + v_n) = \tilde{f} \left( (v_1 + v_2 + \cdots + v_n)^n \right),
\]
there is just one term containing all the vectors \(v_1, v_2, \ldots, v_n\), namely \(n! \cdot \tilde{f}(v_1, v_2, \ldots, v_n)\). For a proper subset \(I \subseteq \{1, 2, \ldots, n\}\), every summand which contains no \(v_i\) with \(i \in I\) appears in (4-27) with the same coefficient as in the expansion (4-27) written for \(f(\sum_{i \in I} v_i)\), because the latter is obtained from \(f(v_1 + v_2 + \cdots + v_n)\) by setting \(v_i = 0\) for all \(i \in I\). Removal of these summands via the standard combinatorial inclusion-exclusion procedure leads to the required formula
\[
n! \cdot \tilde{f}(v_1, v_2, \ldots, v_n) = f \left( \sum_{v} v \right) - \sum_{\{I\}} f \left( \sum_{v \in I} v \right) + \sum_{\{I,j\}} f \left( \sum_{v \in I,j} v \right) - \sum_{\{i,j,k\}} f \left( \sum_{v \neq i,j,k} v \right) + \cdots.
\]

\(^1\)Not necessary finite dimensional.
4.5.1 Duality. For a vector space \( V \) of finite dimension over a field of zero characteristic, the complete contraction between \( V^\otimes m \) and \( V^*\otimes m \) provides the spaces \( S^mV \) and \( S^mV^* \) with the perfect pairing that couples polynomials \( f \in S^nV \) and \( g \in S^nV^* \) to the complete contraction of their complete polarizations \( f' \in V^\otimes m \) and \( g' \in V^*\otimes m \).

Exercise 4.19. For a pair of dual bases \( e_1, e_2, \ldots, e_d \in V, x_1, x_2, \ldots, x_d \in V^* \), verify that all the non-zero couplings between the basis monomials are exhausted by

\[
\langle e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}, x_1 x_2^2 \cdots x_d^{m_d} \rangle = \frac{m_1! m_2! \cdots m_d!}{n!}.
\]

(4-29)

Note that the monomials constructed from the dual basis vectors become the dual bases of the polynomial rings only after rescaling by appropriate combinatorial factors.

4.5.2 Derivative of a polynomial along a vector. Associated with every vector \( v \in V \) is the linear map \( i_v : V^\otimes n \to V^\otimes (n-1), \) \( \varphi \mapsto i_v \varphi, \) provided by the inner multiplication\(^1\) of \( n\)-linear forms on \( V \) by \( v, \) which takes an \( n\)-linear form \( \varphi \in V^\otimes n \) to the \((n-1)\)-linear form

\[
i_v \varphi(v_1, v_2, \ldots, v_{n-1}) = \varphi(v, v_1, v_2, \ldots, v_{n-1}).
\]

Composing this map with preceded complete polarization \( S^nV^* \to Sym^n V^* \subset V^\otimes n \) and subsequent factorization \( \sigma : V^\otimes (n-1) \to S^nV^* \) through the commutativity relations\(^2\), assigns the linear map

\[
pl_v : S^nV^* \to S^{n-1}V^*, \quad f(x) \mapsto pl_v f(x) \equiv \tilde{f}(v, x, x, \ldots, x),
\]

(4-30)

which depends linearly on \( v \in V \). This map fits in the commutative diagram

\[
\begin{array}{ccc}
V^\otimes n & \xrightarrow{i_v} & V^\otimes (n-1) \\
\downarrow{pl} & & \downarrow{\sigma} \\
S^nV^* & \xrightarrow{pl_v} & S^{n-1}V^*.
\end{array}
\]

The polynomial \( pl_v f(x) \tilde{f}(v, x, x, \ldots, x) \in S^{n-1}(V^*) \) is called the polar of \( v \) with respect to \( f \). For \( n = 2 \), the polar of a vector \( v \) with respect to a quadratic worm \( f \in S^2V^* \) is the linear form \( w \mapsto f(v, w) \) considered\(^3\) in 2.2.1 on p. 18.

In terms of dual bases \( e_1, e_2, \ldots, e_d \in V, x_1, x_2, \ldots, x_d \in V^* \), the contraction of the first tensor factor in \( V^\otimes n \) with the basis vector \( e_i \in V \) maps the complete symmetric tensor \( x_{[m_1, m_2, \ldots, m_n]} \) either to the complete symmetric tensor containing the \((m_i - 1)\) factors \( x_i \) or to zero for \( m_i = 0 \). Hence, \( pl_v x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} = m_1! x_1^{m_1} \cdots x_i^{m_i-1} x_{i+1} \cdots x_d^{m_d} = \frac{1}{n!} \sum_{\pi} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \). Since \( pl_v f \) is linear in both \( v, f \), we conclude that for every \( v = \sum a_i e_i \), the polar polynomial of \( v \) with respect to \( f \) is nothing but the derivative of the polynomial \( f \) along the vector \( v \) divided by \( \deg f \), i.e.,

\[
pl_v f = \frac{1}{\deg(f)} \frac{\partial f}{\partial v} = \frac{1}{\deg(f)} \sum_{i=1}^d a_i \frac{\partial f}{\partial x_i}.
\]

\(^1\)See the Example 4.2 on p. 42.

\(^2\)Which is the linear map corresponding to the commutative multiplication of covectors from formula (4-16) on p. 43 by the universal property of tensor product.

\(^3\)Recall that the zero set of this form in \( \mathbb{F}(V) \) is the hyperplane intersecting the quadric \( V(f) \subset \mathbb{F}(V) \) along its apparent contour viewed from \( v \).
Note that this forces the right hand side to be independent on the choice of dual bases in $V$ and $V^*$. It follows from the definition of polar map that the derivatives along vectors commute, $\partial_u \partial_w = \partial_w \partial_u$, and for all $u, w \in V$, $f \in S^n V^*$, $0 \leq m \leq n$, the following relation holds:

$$m! \frac{\partial^m f}{\partial u^m}(w) = n! \tilde{f}(u^m, w^n) = (n - m)! \frac{\partial^{n-m} f}{\partial w^{n-m}}(u). \tag{4.32}$$

**Exercise 4.20.** Prove the Leibniz rule $\partial_v (fg) = \partial_v (f) \cdot g + f \cdot \partial_v (g)$ and show that

$$\tilde{f}(v_1, v_2, \ldots, v_n) = \frac{1}{n!} \partial_{v_1} \partial_{v_2} \ldots \partial_{v_n} f.$$

**Example 4.7 (Taylor’s Expansion)**

For $k = 2$, the expansion (4.27) from the Exercise 4.18 turns to the identity

$$f(u + w) = \tilde{f}(u + w, u + w, \ldots, u + w) = \sum_{m=0}^{n} \binom{n}{m} \tilde{f}(u^m, w^{n-m}),$$

where $n = \deg f$. It holds for any polynomial $f \in S^n V^*$ and all vectors $u, w \in V$. The relations (4.32) allow us to rewrite this identity as the Taylor expansion for $f$ at $u$:

$$f(u + w) = \sum_{m=0}^{\deg f} \frac{1}{m!} \partial_w^m f(u), \tag{4.33}$$

which is an exact equality in the polynomial ring $SV^*$.

**4.5.3 Polars and tangents.** Given a hypersurface $S = V(f) \subset \mathbb{P}(V)$ of degree $n$ and a line $\ell = (pq) \subset \mathbb{P}(V)$, the intersection $\ell \cap S$ consists of all points $\lambda p + \mu q$ such that $(\lambda : \mu) \in \mathbb{P}_1 = \mathbb{P}(\mathbb{k}^2)$ is a root of the homogeneous polynomial $f_{pq}(\lambda, \mu) \equiv f(\lambda p + \mu q) \in \mathbb{k}[\lambda, \mu]$. Over an algebraically closed field $\mathbb{k}$, this polynomial is either zero or a product of $n$ non-zero homogeneous linear forms in $\lambda, \mu$, possibly coinciding:

$$f(\lambda, \mu) = \prod_i (\alpha_i^n \lambda - \alpha_i^\prime \mu)^{s_i} = \prod_i \det_s \left( \begin{array}{cc} \lambda & \alpha_i^n \\ \mu & \alpha_i^\prime \end{array} \right), \tag{4.34}$$

where $\alpha_i = (\alpha_i^n : \alpha_i^\prime)$ are some mutually distinct points on $\mathbb{P}_1$ and $\sum_i s_i = n$. If $f_{pq} = 0$, then $\ell \subset S$. If $f_{pq} \neq 0$, then the intersection $\ell \cap S$ consists of the points $a_i = \alpha_i^n p + \alpha_i^\prime q$. The exponent $s_i$ of the linear form $\alpha_i^n \mu - \alpha_i^\prime \lambda$ in the factorization (4.34) is called the intersection multiplicity of the hypersurface $S$ with the line $\ell$ at the point $a_i$, and is denoted by $(S, \ell)_{a_i}$. If $(S, \ell)_{a_i} = 1$, the intersection point $a_i$ is called simple or transversal. Otherwise, the intersection of $\ell$ and $S$ at $a_i$ is called a multiple. The total number of intersections counted with their multiplicities equals the degree of $S$.

A line $\ell = (pq)$ passing through $p \in S$ is called tangent to $S$ at $p$ if either $\ell \subset S$ or $(S, \ell)_p \geq 2$. In other words, the line $\ell$ is tangent to $S$ at $p$ if the polynomial $f(p + tq) \in \mathbb{k}[t]$ either is the zero polynomial or has a multiple root at zero. The Taylor expansion\(^1\) for $f(p + tq)$ at $p$ starts with

$$f(p + tq) = t \binom{d}{1} \tilde{f}(p^{n-1}, q) + t^2 \binom{d}{2} \tilde{f}(p^{n-2}, q^2) + \ldots.$$

\(^1\)See 4.33 on p. 51.
Therefore the line \( \ell = (pq) \) is tangent to \( S \) at \( p \) if and only if \( \widetilde{f}(p^{n-1}, q) = 0 \). This is the straightforward generalization of the Proposition 2.2 on p. 17.

If \( f(p^{n-1}, x) \) does not vanish identically as a linear form in \( x \), the point \( p \) is called a smooth point of \( S \). The hypersurface \( S \subset \mathbb{P}(V) \) is called smooth if every point \( p \in S \) is smooth. For a smooth \( p \in S \) the linear equation \( F(p^{n-1}, x) = 0 \) on \( x \in V \) defines a hyperplane in \( \mathbb{P}(V) \) filled by the lines \( (pq) \) tangent to \( S \) at \( p \). This hyperplane is called the tangent space to \( S \) at \( p \) and denoted by \( T_p = \{ x \in \mathbb{P}(V) | \widetilde{f}(p^{n-1}, x) = 0 \} \).

If \( f(p^{n-1}, x) \) is the zero linear form in \( x \), the hypersurface \( S \) is called singular at \( p \), and the point \( p \) is called a singular point of \( S \). Since the coefficients of polynomial \( \widetilde{f}(p^{n-1}, x) = \partial_x f(p) \), considered as a linear form in \( x \), are equal to the partial derivatives of \( f \) evaluated at the point \( p \) by (4.32), the singularity of \( p \in S = V(f) \) is expressed by the equations

\[
\frac{\partial f}{\partial x_i}(p) = 0 \quad \text{for all} \; i,
\]

in which case any line \( \ell \) passing through \( p \) has \( (S, \ell)_p \geq 2 \), i.e., is tangent to \( S \) at \( p \). Thus, the tangent lines to \( S \) at a singular point of \( S \) fill the whole ambient space \( \mathbb{P}(V) \).

If \( q \) is either a smooth point on \( S \) or a point outside \( S \), then the polar polynomial

\[
pl_q f(x) = \widetilde{f}(q, x^{n-1})
\]

does not vanish identically as a homogeneous polynomial of degree \( n - 1 \) in \( x \), because otherwise, all partial derivatives of \( pl_q f(x) = \widetilde{f}(q, x^{n-1}) \) in \( x \) would also vanish, and in particular,

\[
\widetilde{f}(q^{n-1}, x) = \frac{\partial^{n-2}}{\partial q^{n-2}} pl_q f(x) = 0
\]

identically in \( x \), meaning that \( q \) is a singular point of \( S \), in contradiction with our choice of \( q \). The zero set of the polar polynomial \( pl_q f \in S^{n-1}V^* \) is denoted by

\[
pl_q S \equiv V(pl_q f) = \{ x \in \mathbb{P}(V) | \widetilde{f}(q, x^{n-1}) = 0 \}
\]

(4.35)

and called the polar hypersurface of the point \( q \) with respect to \( S \). If \( S \) is a quadric, then \( pl_q S \) is exactly the polar hyperplane of \( q \) considered in n’ 2.3.1 on p. 19. As in the Corollary 2.2 on p. 17, for a hypersurface \( S \) of arbitrary degree, the intersection \( S \cap pl_q S \) coincides with the apparent contour of \( S \) viewed from the point \( q \), that is, with the locus of all points \( p \in S \) such that the line \( (pq) \) is tangent to \( S \) at \( p \).

More generally, for an arbitrary point \( q \in \mathbb{P}(V) \) the locus of points

\[
pl_q^{n-r} S \equiv \{ x \in \mathbb{P}(V) | \widetilde{f}(q^{n-r}, x^r) = 0 \}
\]

is called the \( r \)th degree polar of the point \( q \) with respect to \( S \) or the \( r \)th degree polar of \( S \) at \( q \) for \( q \in S \). If the polynomial \( \widetilde{f}(q^{n-r}, x^r) \) vanishes identically in \( x \), we say that the \( r \)th degree polar is degenerate. Otherwise, the \( r \)th degree polar is a projective hypersurface of degree \( r \). The linear\(^1\) polar of \( S \) at a smooth point \( q \in S \) is simply the tangent hyperplane to \( S \) at \( q \): \( pl_q^{n-1} S = T_q S \). The quadratic polar \( pl_q^{n-2} S \) is the quadric passing through \( q \) and having the same tangent hyperplane at \( q \) as \( S \). The cubic polar \( pl_q^{n-3} S \) is the cubic hypersurface passing through \( q \) and having the same quadratic polar at \( q \) as \( S \), etc. The \( r \)th degree polar \( pl_q^{n-r} S \) at a smooth point \( q \in S \) passes through \( q \) and has \( pl_q^{n-k} pl_q^{n-r} S = pl_q^{n-k} S \) for all \( 1 \leq k \leq r - 1 \), because

\[
pl_q^{r-k} pl_q^{n-r} f(x) = \tilde{pl}_q^{n-r} f(q^{n-r}, x^k) = \tilde{f}(q^{n-r}, q^{r-k}, x^k) = f(q^{n-k}, x^k) = pl_q^{n-k} f(x).
\]

\(^1\)That is, of the first degree.
4.5.4 Linear support of a homogeneous polynomial. For a polynomial \( f \in S^n V^* \), we write \( \text{Supp} f \) for the minimal\(^1\) vector subspace \( W \subset V^* \) such that \( f \in S^n W \), and call it the linear support of \( f \). Over a field of zero characteristic, \( \text{Supp} f = \text{Supp} \tilde{f} \), where \( \tilde{f} \in \text{Sym}^n V^* \subset V^{* \otimes n} \) is the complete polarization of \( f \). By the Theorem 4.1, \( \text{Supp} \tilde{f} \) is linearly generated by the images of the \((n-1)\)-tuple contraction maps
\[
c^J_{t} : V^{\otimes(n-1)} \to V^* , \quad t \mapsto c^1,2,\ldots,(n-1)(t \otimes \tilde{f}) ,
\]
coupling all the \((n-1)\) factors of \( V^{\otimes(n-1)} \) with some \( n-1 \) factors of \( \tilde{f} \in V^* \otimes^n \) in order indicated by the sequence \( J = (j_1, j_2, \ldots, j_{n-1}) \). For the symmetric tensor \( \tilde{f} \), such a contraction does not depend on \( J \) and maps every decomposable tensor \( v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1} \) to the linear form on \( V \) proportional to the derivative \( \partial_{v_1} \partial_{v_2} \ldots \partial_{v_{n-1}} \tilde{f} \in V^* \). Thus, \( \text{Supp}(f) \) is linearly generated by all \((n-1)\)-tuple partial derivatives
\[
\frac{\partial^m_1}{\partial x_1^m_1} \frac{\partial^m_2}{\partial x_2^m_2} \ldots \frac{\partial^m_d}{\partial x_d^m_d} f(x) , \quad \text{where} \quad \sum m_j = n-1 . \tag{4-36}
\]
The coefficient of \( x_i \) in the linear form (4-36) depends only on the coefficients of monomial
\[
x_1^{m_1} \ldots x_{i-1}^{m_{i-1}} x_i^{m_i+1} x_{i+1}^{m_{i+1}} \ldots x_d^{m_d}
\]
in \( f \). If we write the polynomial \( f \) as
\[
f = \sum_{v_1,\ldots,v_d=1}^{n!} \frac{n!}{v_1!v_2!\ldots v_d!} a_{v_1v_2\ldots v_d} x_1^{v_1} x_2^{v_2} \ldots x_d^{v_d} , \tag{4-37}
\]
the linear form (4-36) turns to
\[
n! \cdot \sum_{i=1}^{d} a_{m_1 \ldots m_{i-1}(m_i+1)\ldots m_d} x_i . \tag{4-38}
\]
Totally, we get \( {n+d-2 \choose d-1} \) such the linear forms staying in bijection with the nonnegative integer solutions \( m_1, m_2, \ldots, m_d \) of the equation \( m_1 + m_2 + \cdots + m_d = n-1 \).

PROPOSITION 4.3

Let \( k \) be a field of zero characteristic, \( V \) a finite dimensional vector space over \( k \), and \( f \in S^n V^* \) a polynomial written in the form (4-37) in some basis of \( V^* \). If \( f = \psi^n \) for some linear form \( \psi \in V^* \), then the \( d \times \binom{n+d-2}{d-1} \) matrix built from the coefficients of linear forms (4-38) has rank 1. In this case, there are at most \( n \) linear forms \( \varphi \in V^* \) such that \( \psi^n = f \), and they differ from one another by multiplications by the \( n \)th roots of unity laying in \( k \). For algebraically closed field \( k \), the converse is also true: if all the linear forms (4-38) are proportional, then \( f = \psi^n \) for some linear form \( \varphi \) proportional to the forms (4-38).

PROOF. The equality \( f = \psi^n \) means that \( \text{Supp}(f) \subset V^* \) is the 1-dimensional subspace spanned by \( \psi \). In this case, all linear forms (4-38) are proportional to \( \psi \). Such a form \( \psi = \lambda \varphi \) has \( \psi^n = f \) if and only if \( \lambda^n = 1 \in k \). Conversely, let all the linear forms (4-38) be proportional, and \( \psi \neq 0 \) be one of them. Then, \( \text{Supp}(f) = k \cdot \psi \) is the 1-dimensional subspace spanned by \( \psi \). Hence, \( f = \lambda \psi^n \) for some \( \lambda \in k \), and therefore, \( f = \psi^n \) for \( \psi = \sqrt[n]{\lambda} \cdot \psi \). \( \square \)

\(^1\)With respect to inclusions.
\(^2\)Here we use that \( k \) is algebraically closed.
4.5.5 The Veronese varieties \( V(n,k) \). The Veronese map

\[
v_{k,n}: P(V^*) \hookrightarrow P(S^kV^*), \quad \psi \mapsto \psi^n,
\]

(4-39)

for \( \dim V = k + 1 \) embeds \( P_k \) into \( P_n \), where \( N = \binom{n+k}{k} - 1 \). The image of map (4-39) is called the Veronese variety and denoted by \( V(k,n) \subset P(S^nV^*) \). It consists of perfect \( n \) th powers \( \varphi^n \) of linear forms \( \varphi \in V^* \) considered up to proportionality. It follows from the Proposition 4.3 that \( V(n,k) \) is indeed an algebraic projective variety described by a system of quadratic equations asserting the vanishing of all \( 2 \times 2 \)-minors in \( d \times (n^d-k^d-1)^{-1} \) matrix formed by the coefficients of the linear forms (4-38). For example, a homogeneous polynomial in two variables \( f(x_0,x_1) = \sum_{k=0}^n a_k(x_0^{n-k}x_1^k) \) has

\[
\frac{\partial^{n-1} f}{\partial x_0^{n-1} \partial x_1} = n! \cdot (a_1 x_0 + a_{i+1} x_1).
\]

Hence, the image of the Veronese embedding \( v_{i,n}: P_1 \hookrightarrow P_n \) is described by the condition

\[
\text{rk} \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+1} & a_{i+2} & \cdots & a_{i+n} \end{pmatrix} = 1,
\]

which agrees with the Example 1.4 on p. 11 and is equivalent to a system of quadratic equations

\[
\det \begin{pmatrix} a_i & a_j \\ a_{i+1} & a_{j+1} \end{pmatrix} = 0
\]

on the coefficients \( a_i \) of the polynomial \( f \). A polynomial \( f \) satisfies these equations if and only if \( f = \varphi^n \) for some linear form \( \varphi = a_0 x_0 + a_1 x_1 \), and in this case \((a_0 : a_1) = (a_i : a_{i+1})\) for all \( i \).

4.6 Polarization of grassmannian polynomials. The quotient map \( V^\otimes n \to \Lambda^n V \) sends every summand of the basis alternating tensor (4-26)

\[
e_{(i_1,i_2,\ldots,i_n)} \equiv \sum_{g \in S_n} \text{sgn}(g) \cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}}
\]

to the same grassmannian monomial \( e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \). Thus, this map sends \( e_{(i_1,i_2,\ldots,i_n)} \) to \( n! e_I \), and therefore, over a field of zero characteristic, the factorization through the alternating relations assigns the isomorphism \( \text{Alt}^n V \to \Lambda^n V \). By analogy with the usual commutative polynomials, the inverse isomorphism is denoted by \( \text{pl}: \Lambda^n V \to \text{Alt}^n V \) and \( \omega \to \tilde{\omega} \), and called the complete polarization of grassmannian polynomials.

4.6.1 Duality. For a finite dimensional vector space \( V \) over a field of zero characteristic, there is the perfect pairing between the spaces \( \Lambda^n V \) and \( \Lambda^0 V^* \) coupling \( \tau \in \Lambda^n V \) and \( \omega \in \Lambda^0 V^* \) to the complete contraction of their complete polarizations \( \tilde{\tau} \in V^{\otimes n} \) and \( \tilde{\omega} \in V^{\otimes n*} \).  

Exercise 4.21. Convince yourself that the non zero couplings between the basis monomials \( e_I \in \Lambda^n V \) and \( x_J \in \Lambda^0 V^* \) are exhausted by \( \langle e_I, x_J \rangle = 1/n! \).

4.6.2 Partial derivatives in the exterior algebra. Given a covector \( \psi \in V^* \), we write

\[
\text{pl}_\psi: \Lambda^n V \to \Lambda^{n-1} V
\]

for the composition of inner multiplication \( i_\psi: V^{\otimes n} \to V^{\otimes (n-1)} \) by \( \psi \) with preceding complete polarization \( \text{pl}: \Lambda^n V \to \text{Alt}^n V \) and subsequent factorization \( \alpha: V^{\otimes (n-1)} \to \Lambda^{n-1} V \) through the
4.6. Polarization of Grassmannian Polynomials

alternating relations\(^1\). Thus, \(p_\psi\) fits in the commutative diagram

\[
\begin{array}{c}
V^* \otimes V^* \otimes \cdots \otimes V^* \\
\downarrow \quad \downarrow \\
\Lambda^n V^* \\
\end{array}
\]

similar to the diagram from formula (4.31) on p. 50. By analogy with \(n^* 4.5.2\), the polynomial

\[
\partial_\psi \omega \triangleq \deg \omega \cdot p_\psi \omega
\]

is called the derivative of homogeneous Grassmannian polynomial \(\omega \in \Lambda^n V\) in direction of covector \(\psi \in V^*\). Since \(p_\psi \omega\) is linear in \(\psi\), the derivation along \(\psi = \sum \alpha_i x_i\) splits as \(\partial_\psi = \sum \alpha_i \partial_{x_i}\). If \(\omega\) does not depend on \(e_i\), then \(\partial_{x_i} \omega = 0\). Therefore, a nonzero contribution to \(\partial_\psi \omega\) is given only by the derivations \(\partial_{x_i}\) for \(i \in I\).

Exercise 4.22. Check that \(\partial_{x_i} e_i \wedge e_j \wedge \cdots \wedge e_n = e_i \wedge e_j \wedge \cdots \wedge e_n\) for every collection of indexes \(i_1, i_2, \ldots, i_n\), not necessary increasing.

It follows from the Exercise 4.22 that

\[
\partial_{x_i} e_i \wedge e_j \wedge \cdots \wedge e_n = (-1)^{k-1} e_i \wedge e_1 \wedge \cdots \wedge e_1 \wedge e_{k+1} \wedge e_{k+1} \wedge \cdots \wedge e_n
\]

In other words, the derivation of a monomial along the basis covector dual to the \(k\)th variable from the left in the monomial behaves as \((-1)^{k-1} \partial / \partial e_{i_k}\), where the Grassmannian partial derivative \(\partial / \partial e_i\) takes \(e_i\) to 1 and annihilates all \(e_j\) with \(j \neq i\), exactly as in the symmetric case. However, the sign \((-1)^k\) in the previous formula forces the Grassmannian partial derivatives to satisfy the Grassmannian Leibniz rule, which differs from the usual one by an extra sign.

Exercise 4.23 (The Grassmannian Leibniz Rule). For any homogeneous Grassmannian polynomials \(\omega, \tau \in \Lambda V\) and a covector \(\psi \in V\), prove that

\[
\partial_\psi (\omega \wedge \tau) = \partial_\psi \omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge \partial_\psi \tau.
\]

Since the Grassmannian polynomials are linear in each variable, \(\partial^2_\psi \omega = 0\) for all \(\psi \in V\), \(\omega \in \Lambda V\). The relation \(\partial^2_\psi = 0\) forces the Grassmannian derivatives to be super-commutative, that is,

\[
\forall \psi, \xi \in V^* \quad \partial_\psi \partial_\xi = -\partial_\xi \partial_\psi.
\]

4.6.3 Linear Support of a Homogeneous Grassmannian Polynomial. The linear support \(\text{Supp} \omega\) of a homogeneous Grassmannian polynomial \(\omega\) of degree \(n\) is defined to be the minimal\(^2\) vector subspace \(W \subset V\) such that \(\omega \in \Lambda^n W\). It coincides with the linear support of the complete polarization \(\tilde{\omega} \in \text{Skew}^n V\), and is linearly generated by all \((n-1)\)-tuple partial derivatives\(^3\)

\[
\partial_j \omega \triangleq \partial_{x_{j_1}} \partial_{x_{j_2}} \cdots \partial_{x_{j_{n-1}}} \omega = \frac{\partial}{\partial e_{j_1}} \frac{\partial}{\partial e_{j_2}} \cdots \frac{\partial}{\partial e_{j_{n-1}}} \omega,
\]

\(^1\)Which is the linear map corresponding to the alternating multiplication of covectors from formula (4.17) on p. 43 by the universal property of tensor product.

\(^2\)With respect to inclusions.

\(^3\)Compare with \(n^* 4.5.4\) on p. 53.
where \( J = j_1j_2 \ldots j_{n-1} \) runs through all sequences of \( n - 1 \) different indexes taken from the set \{1, 2, \ldots, d\}, \( d = \dim V \). Up to a sign, the order of indexes in \( J \) is not essential, and we will not assume the indexes to be increasing, because this simplifies the notations in what follows.

Let us expand \( \omega \) as a sum of basis monomials

\[
\omega = \sum_l a_l e_l = \sum_{i_1i_2 \ldots i_n} a_{i_1i_2 \ldots i_n} e_{i_1} \land e_{i_2} \land \cdots \land e_{i_n}, \tag{4-42}
\]

where \( l = i_1i_2 \ldots i_n \) also runs through the \( n \)-tuples of different but non necessary increasing indexes, and the coefficients \( a_{i_1i_2 \ldots i_n} \in \mathbb{k} \) are alternating in \( i_1i_2 \ldots i_n \). Nonzero contributions to \( \partial_J \omega \) are given only by the monomials \( a_{i_1} e_j \) with \( l \supset J \). Therefore, up to a common sign,

\[
\partial_J \omega = \pm \sum_{i \not\in J} a_{i_1j_2 \ldots j_{n-1}} e_i. \tag{4-43}
\]

PROPOSITION 4.4
The following conditions on a grassmannian polynomial \( \omega \in \Lambda^n V \) written in the form (4-42) are equivalent:

1) \( \omega = u_1 \land u_2 \land \cdots \land u_n \) for some \( u_1, u_2, \ldots, u_n \in V \)

2) \( u \land \omega = 0 \) for all \( u \in \text{Supp}(\omega) \)

3) for any two collections \( i_1i_2 \ldots i_{m+1} \) and \( j_1j_2 \ldots j_{m-1} \) consisting of \( n + 1 \) and \( n - 1 \) different indexes, the following Plücker relation holds

\[
\sum_{v=1}^{m+1} (-1)^{v-1} a_{j_1 \ldots j_{m-1}i_v} a_{i_1 \ldots i_v \ldots i_{m+1}} = 0, \tag{4-44}
\]

where the hat in \( a_{i_1 \ldots i_v \ldots i_{m+1}} \) means that the index \( i_v \) should be removed.

PROOF. Condition (1) holds if and only if \( \omega \) belongs to the top homogeneous component of its linear span, \( \omega \in \Lambda^{\dim \text{Supp}(\omega)} \text{Supp}(\omega) \). Condition (2) means the same because of the following exercise.

EXERCISE 4.24. Show that \( \omega \in AU \) is homogeneous of degree \( \dim U \) if and only if \( u \land \omega = 0 \) for \( u \in U \).

The Plücker relation (4-44) asserts the vanishing of the coefficient of \( e_{i_1} \land e_{i_2} \land \cdots \land e_{i_{m+1}} \) in the product \( (\partial_{j_1 \ldots j_{m-1}} \omega) \land \omega \). In other words, (4-44) is the coordinate form of condition (2) written for vector \( u = \partial_{j_1 \ldots j_{m-1}} \omega \) from the formula (4-43). Since these vectors linearly generate the subspace \( \text{Supp}(\omega) \), the whole set of the Plücker relations is equivalent to the condition (2).

\[
\square
\]

EXAMPLE 4.8 (THE PLÜCKER QUADRATIC)
Let \( n = 2, \dim V = 4 \), and \( e_1, e_2, e_3, e_4 \) be a basis of \( V \). Then the expansion (4-42) for \( \omega \in \Lambda^2 V \) looks like \( \omega = \sum_{i,j} a_{ij} e_i \land e_j \), where the coefficients \( a_{ij} \) form the alternating \( 4 \times 4 \) matrix. The Plücker relation corresponding to \( (i_1, i_2, i_3) = (2, 3, 4) \) and \( j_1 = 1 \) is

\[
a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0. \tag{4-45}
\]

All other choices of \( (i_1, i_2, i_3) \) and \( j_1 \notin \{i_1, i_2, i_3\} \) lead to exactly the same relation.

EXERCISE 4.25. Check this.

For \( j_1 \in \{i_1, i_2, i_3\} \) we get the trivial equality \( 0 = 0 \). Thus, for \( \dim V = 4 \), the set of decomposable grassmannian quadratic forms \( \omega \in \Lambda^2 V \) is described by just one quadratic equation (5-2).
EXERCISE 4.26. Convince yourself that the equation (5-2) on \( \omega = \sum_{i,j} a_{ij} e_i \wedge e_j \) is equivalent to the condition \( \omega \wedge \omega = 0 \).

4.6.4 The Grasmannian varieties and Plücker embeddings. For a vector space \( V \) of dimension \( d \), the set of all vector subspaces \( U \subset V \) of dimension \( m \) is denoted by \( \text{Gr}(m,V) \) and called the grassmannian. When the origin of \( V \) is not essential or \( V = k^d \), we write \( \text{Gr}(m,d) \) instead of \( \text{Gr}(m,V) \). Thus, \( \text{Gr}(1,V) = \mathbb{P}(V) \), \( \text{Gr}(\dim V - 1, V) = \mathbb{P}(V^*) \). The grassmannian \( \text{Gr}(m,V) \) is embedded into the projective space \( \mathbb{P}(\Lambda^m V) \) by means of the Plücker map

\[
p_m : \text{Gr}(m,V) \to \mathbb{P}(\Lambda^m V), \quad U \mapsto A^m U \subset \Lambda^m V
\]

sending every subspace \( U \subset V \) of dimension \( m \) to its highest exterior power \( \Lambda^m U \), which is a subspace of dimension \( 1 \) in \( \Lambda^m V \). If \( U \) is spanned by vectors \( u_1, u_2, \ldots, u_m \), then up to proportionality, \( p_m(U) = u_1 \wedge u_2 \wedge \cdots \wedge u_m \).

EXERCISE 4.27. Check that the Plücker map is injective.

The image of map (4-46) consists of all grassmannian polynomials \( \omega \in \Lambda^m V \) completely factorisable into a product of \( m \) vectors. Such polynomials are called decomposable. By the Proposition 4.4 they form a projective algebraic variety described by the system of quadratic equations (4-44) on the coefficients of expansion (4-42).

REMARK 4.1. From the algebraic viewpoint, the grassmannian variety \( \text{Gr}(k,m) \subset \mathbb{P}(\Lambda^m V) \) is a super-commutative version of the Veronese variety \( V(k,m) \subset \mathbb{P}(S^m V) \). Both consist of most degenerated non-zero homogeneous polynomials of degree \( m \) in the sense that the linear support of polynomial has the minimal possible dimension which equals \( 1 \) for a commutative polynomial, and equals \( m \) for a grassmannian polynomial of degree \( m \).

EXAMPLE 4.9 (The Grassmannians \( \text{Gr}(2,V) \))

The Plücker embedding identifies the grassmannian \( \text{Gr}(2,V) \) with the set of decomposable grassmannian quadratic forms \( \omega \in \Lambda^2 V \), that is, \( \omega = u \wedge w \) for some \( u, w \in V \). Note that every such \( \omega \) has \( \omega \wedge \omega = u \wedge w \wedge u \wedge w = 0 \). For an arbitrary \( \omega \in \Lambda^2 V \), there exists a basis \( \xi_1, \xi_2, \ldots, \xi_d \) in \( V \) such that \( \omega = \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots \). If this sum contains more than one term, then the monomial \( \xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 \) appears in \( \omega \wedge \omega \) with the coefficient 2 and therefore, \( \omega \wedge \omega \neq 0 \). Thus, such \( \omega \) is not decomposable. We conclude that \( \omega \in \Lambda^2 V \) is decomposable if and only if \( \omega \wedge \omega = 0 \).

For \( \dim V = 4 \), the squares of forms \( \omega \in \Lambda^2 V \) lie in the space \( \Lambda^4 V \) of dimension \( 1 \). In this case, the condition \( \omega \wedge \omega = 0 \) for \( \omega = \sum_{i,j} a_{ij} e_i \wedge e_j \) is expressed by just one quadratic equation

\[
a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0,
\]

which agrees with the equation (5-2) from the Example 4.8 on p. 56. We conclude that the Plücker embedding identifies the grassmannian \( \text{Gr}(2, V) \) with the quadric (4-47) in \( \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V) \). This quadric is called the Plücker quadric.

EXAMPLE 4.10 (The Segre varieties revisited\(^2\))

Let \( W = V_1 \oplus V_2 \oplus \cdots \oplus V_n \) be a direct sum of finite dimensional vector spaces \( V_i \). For every collection of non-negative integers \( m_1, m_2, \ldots, m_n \) such that \( m_i \leq \dim V_i \), put \( k = \sum m_i \) and

\(^1\)See the Example 4.5 on p. 47.

\(^2\)See n’4.1.2 on p. 39.
denote by \( W_{m_1, m_2, ..., m_n} \subset \mathcal{A}^k W \) the linear span of all products \( w_1 \wedge w_2 \wedge \cdots \wedge w_k \) formed by \( m_1 \) vectors taken from \( V_1 \), \( m_2 \) vectors taken from \( V_2 \), etc.

**Exercise 4.28.** Show that the well defined isomorphism of vector spaces

\[
A^{m_1} V_1 \otimes A^{m_2} V_2 \otimes \cdots \otimes A^{m_n} V_n \cong W_{m_1, m_2, ..., m_n}
\]

is assigned by prescription \( \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n \mapsto \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \), and verify that

\[
\mathcal{A}^k W = \bigoplus_{m_1, m_2, ..., m_n} W_{m_1, m_2, ..., m_n} \cong \bigoplus_{m_1, m_2, ..., m_n} A^{m_1} V_1 \otimes A^{m_2} V_2 \otimes \cdots \otimes A^{m_n} V_n.
\]

We conclude that the tensor product \( V_1 \otimes V_2 \otimes \cdots \otimes V_n \) can be identified with the component \( W_{1,1,1} \subset \mathcal{A}^n W \). Under this identification, the decomposable tensors \( v_1 \otimes v_2 \otimes \cdots \otimes v_n \) go to the decomposable Grassmannian monomials \( v_1 \wedge v_2 \wedge \cdots \wedge v_n \). Therefore, the Segre variety from 4.1.2 on p. 39 is the intersection of the Grassmannian variety \( \text{Gr}(n, W) \subset \mathbb{P}(\mathcal{A}^n W) \) with the projective subspace \( \mathbb{P}(W_{1,1,1}) \subset \mathbb{P}(\mathcal{A}^n W) \). In particular, the Segre variety is indeed an algebraic variety described by the system of quadratic equations from the Proposition 4.4 on p. 56 restricted onto the linear subspace \( W_{1,1,1} \subset \mathcal{A}^n W \).
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EXRC. 4.3. The first statement is verified by the same arguments as in ?? on p. ?? and n’ 2.5.1. To prove the second, chose some dual bases $u_1, u_2, \ldots, u_n \in U$, $u'_1, u'_2, \ldots, u'_n \in U^*$ and a basis $w_1, w_2, \ldots, w_m \in W$. Then $mn$ decomposable tensors $u'_i \otimes w_j$ form a basis in $U^* \otimes V$. The matrix of operator

$$u'_i \otimes w_j : u_k \mapsto \begin{cases} w_j & \text{for } k = i \\ 0 & \text{otherwise} \end{cases}$$

has 1 in the crossing of $j$ th row with $i$ th column and zeros elsewhere. Thus, these operators span $\text{Hom}(U, W)$.

EXRC. 4.4. For any linear mapping $f : V \to A$ the multiplication

$$V \times V \times \cdots \times V \to A,$$

which takes $(v_1, v_2, \ldots, v_n)$ to their product $\varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n) \in A$, is multilinear. Hence, for each $n \in \mathbb{N}$ there exists a unique linear mapping $V^\otimes n \to A$ taking tensor multiplication to multiplication in $A$. Add them all together and get required algebra homomorphism $TV \to A$ extending $f$. Since any algebra homomorphism $TV \to A$ that extends $f$ has to take $v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n)$, it coincides with the extension just constructed. Uniqueness of free algebra is proved exactly like the Lemma 4.1 on p. 39.

EXRC. 4.5. Since the decomposable tensors span $V^\otimes n$ and the equality

$$i_v \varphi(w_1, w_2, \ldots, w_{n-1}) = \varphi(v, w_1, w_2, \ldots, w_{n-1})$$

is bilinear in $v$, $\varphi$, it is enough to check it for the decomposable $\varphi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$.

EXRC. 4.6. Fix a basis $e_1, \ldots, e_p, u_1, \ldots, u_q, w_1, \ldots, w_r, v_1, \ldots, v_s$ in $V$ such that $e_i$ form a basis in $U \cap W$, $u_i$ and $w_i$ extend it to some bases in $U$, $W$, and $v_m$ complete everything to a basis in $V$. Then expand $t$ through the standard monomial basis of $TV$ built from this basis of $V$.

EXRC. 4.8. For all $v, w \in V$ we have

$$0 = \varphi(..., (v + w), ..., (v + w), ...) = \varphi(..., v, ..., w, ..., v, ..., v, ...) + \varphi(..., v, ..., v, ..., v, ..., v, ...).$$

Vice versa, if $\text{char } k \neq 2$, then $\varphi(..., v, ..., v, ...) = -\varphi(..., v, ..., v, ..., v, ...)$. Thus, $\varphi(..., v, ..., v, ...) = 0$.


EXRC. 4.10. Every multilinear map $\varphi : V \times V \times \cdots \times V \to W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $F : V^\otimes n \to W$ is linear. Such $F$ is factorized through the projection $V^\otimes n \to S^n V$ if and only if

$$F(... \otimes v \otimes w \otimes \cdots) = F(... \otimes w \otimes v \otimes \cdots).$$

The latter is equivalent to $\varphi(..., v, w, ...) = \varphi(..., w, v, ...)$. This proves the universality of the multiplication in $SV$. Every linear map $f : V \to A$ induces the symmetric multilinear map $V \times V \times \cdots \times V \to A, (v_1, v_2, \ldots, v_n) \mapsto \prod \varphi(v_i)$ for any $n \in \mathbb{N}$. The latter gives the linear map
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\[ S^n V \to A. \] All together these maps extend \( f \) to the homomorphism of \( A \)-algebras \( SV \to A \). Vice versa, every homomorphism of \( A \)-algebras \( SV \to A \), which extends \( f \), takes \( \prod v_i \to \prod \phi(v_i) \) and coincides with the previous extension. The uniqueness of extension is verified as in the Lemma 4.1 on p. 39.

**Exrc. 4.11.** The first follows from \( 0 = (v + w) \otimes (v + w) = v \otimes w + w \otimes v \), the second from \( v \otimes v + v \otimes v = 0 \).

**Exrc. 4.12.** Similar to ?? on p. ??.

**Exrc. 4.13.** If \( \dim V = d \), then \( Z(AV) = A^d V + \bigoplus_k A^{2k} V \). For even \( d \), the first summand is contained in the second, for odd \( d \) the sum is direct.

**Exrc. 4.14.** Use that \( \det A = \det A^t \), and transpose everything.

**Exrc. 4.15.** The summands form one \( S_n \)-orbit. The stabilizer of an element in this orbit consists of \( m_1! \cdot m_2! \cdots m_d! \) independent permutations of coinciding factors. Hence, the length of orbit equals \( m_1! \cdot m_2! \cdots m_d! \).

**Exrc. 4.16.** For \( v = \sum a_i e_i \), the complete contraction of \( v^{\otimes n} \) with \( \tilde{\alpha} = \frac{m_1! \cdot m_2! \cdots m_d!}{n!} x_{[m_1, m_2, \ldots, m_d]} \) is the sum of \( n!/(m_1! \cdot m_2! \cdots m_d!) \) mutually equal products

\[ \frac{m_1! \cdot m_2! \cdots m_d!}{n!} x_{[m_1, m_2, \ldots, m_d]} = \frac{m_1! \cdot m_2! \cdots m_d!}{n!} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}. \]

Thus, it coincides with the result of substitution \((x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)\) in the monomial \( m_1! \cdot m_2! \cdots m_d! x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \).

**Exrc. 4.17.** Use the same arguments as in the proof of multinomial expansion formula

\[ (v_1 + v_2 + \cdots + v_k)^n = \sum_{m_1, m_2, \ldots, m_k} \frac{n!}{m_1! m_2! \cdots m_k!} v_1^{m_1} v_2^{m_2} \cdots v_k^{m_k}. \]

**Exrc. 4.18.** Since the Leibniz rule is linear in \( v, f, g \), it is enough to check it for \( v = e_i \), \( f = x_1^{m_1} \cdots x_d^{m_d} \), \( g = x_i^k \). In this case it follows directly from the definition of polar map. The formula for \( f(v_1, v_2, \ldots, v_k) \) follows from the equality \( f(v_1, x, \ldots, x) = \frac{1}{n} \cdot \partial_v f(x) \) by induction in \( n = \deg f \).

**Exrc. 4.19.** Similar to the Exercise 4.20.

**Exrc. 4.20.** Let \( e_1, e_2, \ldots, e_m \) be a basis in \( U \). If \( \omega \notin A^m U \), then the expansion of \( \omega \) as a linear combination of basis monomials \( e_l \) contains a monomial whose index \( l \) differs from the whole \( 1, 2, \ldots, m \). Let \( k \neq l \). Then \( e_k \land \omega \neq 0 \), because the basis monomial \( e^{[k]} \) appears in \( e_k \land \omega \) with a nonzero coefficient. Conversely, if \( \omega \in A^m U \), then \( \omega = \lambda \cdot e_1 \land e_2 \land \cdots \land e_m \) and \( e_i \land \omega = 0 \) for all \( i \).

**Exrc. 4.21.** See the Example 4.9 on p. 57.

**Exrc. 4.22.** Let \( U \neq W \) be two subspaces of dimension \( m \). Chose a basis \( e_1, e_2, \ldots, e_r, u_1, u_2, \ldots, u_{m-r}, w_1, w_2, \ldots, w_{m-r}, v_1, v_2, \ldots, v_{d+r-2m} \in V \) such that \( e_1, e_2, \ldots, e_r \) is a basis of \( U \cap W \), vectors \( u_1, u_2, \ldots, u_{m-r} \) and \( w_1, w_2, \ldots, w_{m-r} \) complete it to bases in \( U \) and \( W \) respectively, and the remaining vectors are complementary to \( U + W \). The Plücker embedding (???) sends \( U \) and \( V \) to the different basis monomials

\[ v_1 \land \cdots \land v_r \land u_1 \land \cdots \land u_{m-r} \neq v_1 \land \cdots \land v_r \land w_1 \land \cdots \land w_{m-r} \]

in \( A^m V \).