§4 Tensor Guide

4.1 Tensor products and Segre varieties. Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over a field $\Bbbk$. A map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ is called *multilinear*, if it is linear in each argument when all the other are fixed:

$$\varphi(\dots, \lambda v' + \mu v'', \dots) = \lambda \varphi(\dots, v', \dots) + \mu \varphi(\dots, v'', \dots).$$

Multilinear maps $V_1 \times V_2 \times \cdots \times V_n \to W$ form a vector space denoted $\operatorname{Hom}(V_1, V_2, \dots, V_n; W)$. As soon some bases $e_1, e_2, \dots, e_m \in W$ and $e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)} \in V_i$, $1 \leqslant i \leqslant n$, are fixed, every multilinear map $\varphi \in \operatorname{Hom}(V_1, V_2, \dots, V_n; W)$ can be uniquely described by the values on all collections of basis vectors:

$$\varphi\left(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)}\right) = \sum_\nu a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot e_\nu \in W,$$

that is, by $m \cdot \prod d_\nu$ constants $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \in \Bbbk$, which can be organized in the matrix of dimension $(n+1)$ and size[1] $m \times d_1 \times d_2 \times \cdots \times d_n$. The multilinear map φ corresponding to such a matrix sends a collection of vectors $v_1, v_2, \dots, v_n$, where $v_i = \sum_{\alpha_i=1}^{d_i} x_{\alpha_i}^{(i)} e_{\alpha_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, to the vector

$$\varphi(v_1, v_2, \dots, v_n) = \sum_{\nu=1}^{m} \left(\sum_{\alpha_1, \alpha_2, \dots, \alpha_n} a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot x_{\alpha_1}^{(1)} \cdot x_{\alpha_2}^{(2)} \cdot \cdots \cdot x_{\alpha_n}^{(n)} \right) \cdot e_\nu \in W.$$

Thus, $\dim \operatorname{Hom}(V_1, V_2, \dots, V_n; W) = \dim W \cdot \prod_\nu \dim V_\nu$.

Exercise 4.1. Check that A) a collection of vectors $v_1, v_2, \dots, v_n \in V_1 \times V_2 \times \cdots \times V_n$ does not contain the zero vector if and only if there exists a multilinear map φ such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$ B) for a linear $F : U \to W$ and multilinear $\varphi : V_1 \times V_2 \times \cdots \times V_n \to U$, the composition $F \circ \varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ is multilinear.

4.1.1 Tensor product of vector spaces. Given a multilinear map

$$\tau : V_1 \times V_2 \times \cdots \times V_n \to U \tag{4-1}$$

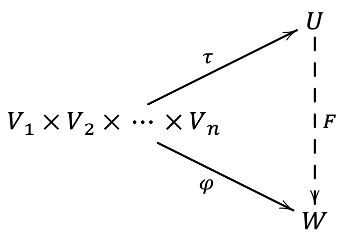
and a vector space W, composing τ with linear maps $F : U \to W$ assigns the map

$$\operatorname{Hom}(U, W) \xrightarrow{F \mapsto F \circ \tau} \operatorname{Hom}(V_1, V_2, \dots, V_n; W) \tag{4-2}$$

which is obviously linear in F.

Definition 4.1

A multilinear map (4-1) is called *universal* if for any vector space W, the linear map (4-2) is an isomorphism. In the expanded form, this means that for every vector space W and multilinear map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$, there exist a unique linear operator $F : U \to W$ such that $\varphi = F \circ \tau$, i.e., two solid multilinear arrows in the diagram



[1] The usual matrices of dimension 2 and size $d \times m$ describe *linear* maps $V \to W$.

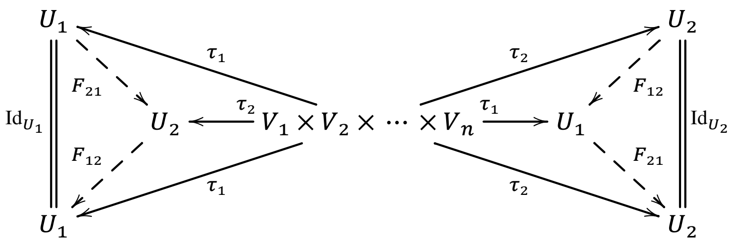
are uniquely completed to a commutative triangle by the dashed linear arrow.

LEMMA 4.1
For every two universal multilinear maps

$$\tau_1 : V_1 \times V_2 \times \cdots \times V_n \to U_1, \quad \tau_2 : V_1 \times V_2 \times \cdots \times V_n \to U_2,$$

there exists a unique linear isomorphism $\iota : U_1 \xrightarrow{\sim} U_2$ such that $\tau_2 = \iota\tau_1$.

PROOF. By the universal properties of τ_1, τ_2, there exists a unique pair of linear maps $F_{21} : U_1 \to U_2$ and $F_{12} : U_2 \to U_1$ that fit in the commutative diagram



Since the factorizations $\tau_1 = \varphi \circ \tau_1$, $\tau_2 = \psi \circ \tau_2$ are unique and hold for $\varphi = \mathrm{Id}_{U_1}$, $\psi = \mathrm{Id}_{U_2}$, we conclude that $F_{21}F_{12} = \mathrm{Id}_{U_2}$ and $F_{12}F_{21} = \mathrm{Id}_{U_1}$. □

LEMMA 4.2
Given a basis $e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, write $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for the vector space with basis formed by $\prod d_i$ formal expressions

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}, \quad 1 \leqslant \alpha_i \leqslant d_i. \tag{4-3}$$

Then the multilinear map $\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$ sending every collection of basis vectors $\left(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \ldots, e_{\alpha_n}^{(n)}\right) \in V_1 \times V_2 \times \cdots \times V_n$ to the expression (4-3) is universal.

PROOF. For a multilinear $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ and linear $F : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W$, the identity $\varphi = F \circ \tau$ mans exactly that $F\left(e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}\right) = \varphi\left(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \ldots, e_{\alpha_n}^{(n)}\right)$ for all collections of basis vectors. □

DEFINITION 4.2
The universal multilinear map (4-1) is denoted by

$$\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n, \quad (v_1, v_2, \dots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n \tag{4-4}$$

and called *tensor multiplication*. The target space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called the *tensor product* of spaces $V_1, V_2, \dots, V_n$ and its elements are called *tensors*.

4.1.2 Decomposable tensors and Segre varieties. The image of tensor multiplication (4-4) consists of the tensor products $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ called *tensor monomials* or *decomposable tensors*. They do not form a vector space, because the map (4-4) is not linear but multilinear. However, the linear span of decomposable tensors is the whole space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Over an infinite ground field, a random tensor is most likely an indecomposable linear combination of tensor monomials.

Geometrically, the tensor multiplication assigns a map

$$s : \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \to \mathbb{P}\left(V_1 \otimes V_2 \otimes \cdots \otimes V_n\right) \tag{4-5}$$

sending a collection of dimension 1 subspaces $\Bbbk \cdot v_i \subset V_i$ spanned by non zero vectors $v_i \in V_i$ to the dimension 1 subspace $\Bbbk \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_n \subset V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

EXERCISE 4.2. Verify that the map (4-5) is a well defined and injective.

The map (4-5) is called the *Segre embedding* and its image, i.e., the projectivization of the set of decomposable tensors, is called the *Segre variety*. Since the decomposable tensors linearly span the whole space, the Segre variety is not contained in a hyperplane. Note that the dimension of Segre variety equals $\sum m_i$, where $m_i = d_i - 1$, and is much smaller then $\dim \mathbb{P}\left(V_1 \otimes V_2 \otimes \cdots \otimes V_n\right) = = \prod(1+m_i)-1$. By the construction, the Segre variety is ruled by n families of projective subspaces of dimensions $m_1, m_2, \dots, m_n$. The simplest example of the Segre variety is provided by the Segre quadric from n° 2.5.1 on p. 22.

EXAMPLE 4.1 (DECOMPOSABLE LINEAR MAPS)
For any two vector spaces U, W, the bilinear map $U^* \times W \to \operatorname{Hom}(U, V)$ is provided by sending $(\xi, w) \in U^* \times W$ to the linear operator $U \to W$, $u \mapsto \langle\, \xi\,,\, u\,\rangle \cdot w$. By the universal property of tensor multiplication, there exists a unique linear map

$$U^* \otimes V \to \operatorname{Hom}(U, V) \tag{4-6}$$

sending every decomposable tensor $\xi \otimes w$ to the same operator. Note that this operator has rank 1, its image is spanned by $w \in W$, and the kernel is $\operatorname{Ann}(\xi) \subset U$.

EXERCISE 4.3. Check that A) every linear map $F : U \to W$ of rank 1 equals $\xi \otimes w$ for appropriate $\xi \in U^*$, $w \in W$ uniquely up to proportionality determined by F B) the linear map (4-6) is an isomorphism for any vector spaces U and V of finite dimensions.

Geometrically, the operators of rank 1 form the Segre variety $S \subset \mathbb{P}_{mn-1} = \mathbb{P}(\operatorname{Hom}(U, W))$, which is ruled by two families of projective spaces $\xi \otimes \mathbb{P}(W)$, $\mathbb{P}(U^*) \otimes w$ and is not contained in a hyperplane. If we fix some bases in U, W, write operators $U \to W$ by their matrices $A = \left(a_{ij}\right)$ in these bases, and use the matrix elements a_{ij} as the homogeneous coordinates in $\mathbb{P}(\operatorname{Hom}(V, W))$, then the Segre variety is described by the equation $\operatorname{rk} A = 1$, which encodes the system of homogeneous quadratic equations

$$\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{\ell j} a_{\ell k} - a_{ik} a_{\ell j} = 0$$

for all $1 \leqslant i < \ell \leqslant \dim W$, $1 \leqslant j < k \leqslant \dim U$. The Segre embedding

$$\mathbb{P}(U^*) \times \mathbb{P}(V) = \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \hookrightarrow \mathbb{P}_{mn-1} = \mathbb{P}(\operatorname{Hom}(U, W))$$

takes a pair of points $x = (x_1 : x_2 : \cdots : x_n)$, $y = (y_1 : y_2 : \cdots : y_n)$ to the rank 1 matrix $A(x, y) = y^t \cdot x$ whose $a_{ij} = x_j y_i$. For $\dim U = \dim W = 2$, we get the Segre quadric in $\mathbb{P}_3$ discussed in n° 2.5.1 on p. 22.

4.2 Tensor algebra and contractions. Given a vector space V, we write $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ for the tensor product of n copies of V an call it the n th *tensor power* of V. We also put $V^{\otimes 0} \stackrel{\text{def}}{=} \Bbbk$, $V^{\otimes 1} \stackrel{\text{def}}{=} V$. The infinite direct sum $\mathsf{T}V \stackrel{\text{def}}{=} \bigoplus_{n \geqslant 0} V^{\otimes n}$ is called the *tensor algebra* of V. This is an associative (non-commutative) graded algebra with the multiplication provided by the tensor product of vectors. For every basis $e_1, e_2, \dots, e_n$ in V, the tensor monomials

$$e_{\nu_1} \otimes e_{\nu_2} \otimes \cdots \otimes e_{\nu_m} \tag{4-7}$$

form a basis of $\mathsf{T}V$ over $\Bbbk$. These monomials are multiplied just by writing them sequentially with the sign $\otimes$ between then. Linear combinations of monomials are multiplied by the usual distributivity rules. Thus, $\mathsf{T}V$ may be thought of as the algebra of polynomials in n non-commuting variables e_ν. Another name for $\mathsf{T}V$ is the *free associative* $\Bbbk$*-algebra with unit* spanned by the vector space V. This name emphasizes the following universal property of the $\Bbbk$-linear map

$$\iota : V \hookrightarrow \mathsf{T}V \tag{4-8}$$

embedding V into $\mathsf{T}V$ as the subspace $V^{\otimes 1}$ of linear homogeneous polynomials.

EXERCISE 4.4. Prove that for every associative $\Bbbk$-algebra A with unit and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of associative $\Bbbk$-algebras $\alpha : \mathsf{T}V \to A$ such that[1] $f = \alpha \circ \iota$. Convince yourself that this property characterizes the inclusion (4-8) uniquely up to a unique isomorphism of the target space commuting with the inclusion.

4.2.1 Total contraction and duality. There is the canonical pairing between $(V^*)^{\otimes n}$ and $V^{\otimes n}$ provided by the *total contraction*, which sends $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$, $v = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ to

$$\langle\, \xi\,,\, v\,\rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n} \langle\, \xi_i\,,\, v_i\,\rangle\,. \tag{4-9}$$

Since the right hand side is multilinear in v_i's, every collection of ξ_i's assigns the well defined linear map $V^{\otimes n} \to \Bbbk$, which depends on ξ_i's also multilinearly. Hence, the contraction of decomposable tensors (4-9) is uniquely extended to the bilinear pairing $V^{*\otimes n} \times V^{\otimes n} \to \Bbbk$. For a pair of dual bases $e_1, e_2, \dots, e_n \in V$, $x_1, x_2, \dots, x_n \in V^*$, the tensor monomials $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ and $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_s}$ form the dual bases of $\mathsf{T}V$ and $\mathsf{T}V^*$ with respect to this pairing. In particular, for a finite dimensional vector space V, we have the canonical isomorphism

$$\left(V^{\otimes n}\right)^* \simeq (V^*)^{\otimes n}\,. \tag{4-10}$$

It follows from the universal property of $V^{\otimes n}$ that the space $\left(V^{\otimes n}\right)^*$ of the linear maps $V^{\otimes n} \to \Bbbk$ is canonically isomorphic to the space of multilinear maps $V \times V \times \cdots \times V \to \Bbbk$, i.e.,

$$\left(V^{\otimes n}\right)^* \simeq \operatorname{Hom}(V, \dots, V\,;\, \Bbbk)\,. \tag{4-11}$$

Combining (4-10) and (4-11) leads to the canonical isomorphism

$$(V^*)^{\otimes n} \simeq \operatorname{Hom}(V, \dots, V\,;\, \Bbbk)\,. \tag{4-12}$$

It sends a decomposable tensor $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$ to the multilinear map $V \times V \times \cdots \times V \to \Bbbk$ taking $(v_1, v_2, \dots, v_n) \mapsto \prod_{i=1}^{n} \xi_i(v_i)$.

[1] In other words, for every $\Bbbk$-algebra A, the homomorphisms of $\Bbbk$-algebras $\mathsf{T}V \to A$ stay in bijection with the $\Bbbk$-linear maps $V \to A$.

4.2.2 Partial contractions. Consider two inclusions[1] of sets

$$\{1, 2, \dots, p\} \xleftarrow{I} \{1, 2, \dots, m\} \xrightarrow{J} \{1, 2, \dots, q\},$$

and write i_ν, j_ν for $I(\nu)$, $J(\nu)$ respectively. Thus, we have two numbered collections of indexes $I = (i_1, i_2, \dots, i_m)$, $J = (j_1, j_2, \dots, j_m)$ staying in the fixed bijection. A *partial contraction* of $V^{*\otimes p}$ and $V^{\otimes q}$ in indexes I, J is the linear map

$$c^I_J : V^{*\otimes p} \otimes V^{\otimes q} \to V^{*\otimes(p-m)} \otimes V^{\otimes(q-m)}$$

which contracts i_ν th factor of $V^{*\otimes p}$ with j_ν th factor of $V^{\otimes q}$ for every $\nu = 1, 2, \dots, m$ and keeps all the other factors in their initial order:

$$\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_q \mapsto \prod_{\nu=1}^m \langle \xi_{i_\nu}, v_{j_\nu} \rangle \cdot \big(\bigotimes_{i\notin I} \xi_i\big) \otimes \big(\bigotimes_{j\notin J} v_j\big). \tag{4-13}$$

Note that different choices of the maps I, J lead to the different contraction maps even if the images of I, J remain unchanged.

Example 4.2 (innner product between vectors and multilinear forms)
Let us treat a n-linear form $\varphi(v_1, v_2, \dots, v_n)$ as a tensor from $V^{*\otimes n}$ via isomorphism (4-12). The contraction of this tensor with a vector $v \in V$ in the first tensor factor is a tensor from $V^{*\otimes(n-1)}$, which can be considered as an $(n-1)$-linear form on V. This form is called the *inner product* of v and φ and denoted by $i_v\varphi$ or $v \llcorner \varphi$.

Exercise 4.5. Check that $i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v, w_1, w_2, \dots, w_{n-1})$.

4.2.3 The linear support of a tensor. Given a tensor $t \in V^{\otimes n}$, the intersection of all vector subspaces $W \subset V$ such that $t \in W^{\otimes n}$ is called the *linear support* of t and denoted by $\mathrm{Supp}(t) \subset V$. It follows from the next the Exercise 4.6 that $\mathrm{Supp}(t)$ is the unique minimal[2] subspace in V among those $W \subset V$ for which $t \in W^{\otimes n}$.

Exercise 4.6. For any subspaces $U, W \subset V$, verify that $U^{\otimes n} \cap W^{\otimes n} = (U \cap W)^{\otimes n}$ in $V^{\otimes n}$.

The dimension of $\mathrm{Supp}\, t$ is called the *rank* of t and denoted by $\mathrm{rk}\, t \overset{\text{def}}{=} \dim \mathrm{Supp}\, t$. We say that t is *degenerated* if $\mathrm{rk}\, t < \dim V$. In this case, the number of variables in the expansion of t through the basis tensor monomials can be reduced by a linear change of variables.

Exercise 4.7. Show that if $\dim \mathrm{Supp}(t) = 1$ and the ground field is algebraically closed, then $t = \lambda \cdot v^{\otimes n}$ for some $\lambda \in \Bbbk$, $v \in V$.

The space $\mathrm{Supp}(t)$ admits an effective description as a linear span of some finite collection of vectors constructed by means of contraction maps. Namely, for every injective[3] map

$$J : \{1, 2, \dots, (n-1)\} \hookrightarrow \{1, 2, \dots, n\}, \tag{4-14}$$

write $\{j_1, j_2, \dots, j_{n-1}\} \subset \{1, 2, \dots, n\}$ for the image of J and $\hat{j}$ for the remaining index outside $\mathrm{im}\, J$. Consider the contraction map

$$c^J_t : V^{*\otimes(n-1)} \to V, \quad \xi \mapsto c^{(1, 2, \dots, (n-1))}_{(j_1, j_2, \dots, j_{n-1})}(\xi \otimes t) \tag{4-15}$$

[1] Not necessary monotonous.
[2] With respect to inclusions.
[3] Not necessary monotonous.

which couples ν th tensor factor of $V^{*\otimes(n-1)}$ with j_ν th tensor factor of t for all $1 \leqslant \nu \leqslant (n-1)$. The result of such contraction is obviously a linear combination of $\hat{j}$ th tensor factors of t. Thus, it belongs to $\mathrm{Supp}(t)$.

THEOREM 4.1
For every $t \in V^{\otimes n}$, the linear support $\mathrm{Supp}(t) \subset V$ is spanned by the images of all contraction maps (4-15) coming from $n!$ different choices of the map (4-14).

PROOF. Let $\mathrm{Supp}(t) = W \subset V$. It is enough to check that every linear form $\xi \in V^*$ annihilating all the subspaces $\mathrm{im}\left(c_t^J\right)$ annihilates W as well. Assume the contrary: let a linear form $\xi \in V^*$ annihilate all $c_t^J\left(V^{*\otimes(n-1)}\right)$ but have a non-zero restriction on W. Chose a basis $\xi_1, \xi_2, \dots, \xi_d \in V^*$ such that $\xi_1 = \xi$ and the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ on W form a basis in W^*. Expand t through the tensor monomials built from the dual basis vectors $w_1, w_2, \dots, w_k \in W$. The value

$$\xi\left(c_t^J(\xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}})\right)$$

is equal to the complete contraction of t with the basic monomial $\xi_1 \otimes \xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}}$ in the order of coupling prescribed by J. This contraction kills all tensor monomials in the expansion of t except for the one, dual to the monomial obtained from $\xi_1 \otimes \xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}}$ by some permutation of factors depending on J. Thus, the result of contraction is equal to the coefficient of some monomial containing w_1 in the expansion of t. Since every such monomial can be reached by appropriate choice of J, we conclude that $w_1 \notin \mathrm{Supp}(t)$. Contradiction. □

4.3 Symmetric and grassmannian algebras. A multilinear map $\varphi : V \times V \times \cdots \times V \to U$ is called *symmetric* if it remains unchanged under permutations of the arguments, and *alternating* if it vanishes as soon some of the arguments coincide.

EXERCISE 4.8. Verify that under a permutation of the arguments, the value of an alternating multilinear map is multiplied by the sign of permutation. Convince yourself that this property implies the alternating property if $\mathrm{char}\,\Bbbk \neq 2$.

We write $\mathrm{Sym}^n(V, U) \subset \mathrm{Hom}(V, \dots, V; U)$ and $\mathrm{Alt}^n(V, U) \subset \mathrm{Hom}(V, \dots, V; U)$ for subspaces of symmetric and alternating multilinear maps. Everything said about the universal multilinear maps in n° 4.1.1 on p. 38 makes sense separately for the symmetric and alternating maps as well. The universal symmetric multilinear map is denoted by

$$\sigma : V \times V \times \cdots \times V \to S^n V, \quad (v_1, v_2, \dots, v_n) \mapsto v_1 v_2 \dots v_n, \tag{4-16}$$

and called the *commutative* multiplication of vectors. Its target space $S^n V$ is called the n th *symmetric power* of V. The universal alternating multilinear map is denoted by

$$\alpha : V \times V \times \cdots \times V \to \Lambda^n V, \quad (v_1, v_2, \dots, v_n) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_n, \tag{4-17}$$

and called the *exterior*[1] multiplication of vectors. Its target space $\Lambda^n V$ is called the n th *exterior power* of V. The universal symmetric and alternating multilinear maps are unique up to a unique isomorphism of the target space commuting with the universal map. The both can be constructed for all n at once by factorizing the tensor algebra $\mathsf{T}V$ by appropriate two-sided ideals.

[1]Also known as *grassmannian* or *super-commutative*.

4.3.1 The symmetric algebra. Write $I_{\text{com}} \subset \mathsf{T}V$ for a two-sided ideal spanned by all the differences

$$u \otimes w - w \otimes u\,, \quad u, w \in V\,. \tag{4-18}$$

This ideal is obviously homogeneous in the sense that $I_{\text{com}} = \bigoplus_{n\geqslant 0}\left(I_{\text{com}} \cap V^{\otimes n}\right)$, and the degree n component $I_{\text{com}} \cap V^{\otimes n}$ of I_{com} is linearly generated over $\Bbbk$ by all differences of the form

$$(\cdots \otimes v \otimes w \otimes \cdots) - (\cdots \otimes w \otimes v \otimes \cdots)\,, \tag{4-19}$$

where the both terms are decomposable of degree n and vary only in the order of v, w. The factor algebra $SV \overset{\text{def}}{=} \mathsf{T}V / I_{\text{com}}$ is called the *symmetric algebra* of V. The multiplication in SV comes from the tensor multiplication in $\mathsf{T}V$ and is commutative, because of the relations $uw = wu$ appearing after the factorization through (4-18). The symmetric algebra is graded

$$SV = \bigoplus_{n\geqslant 0} S^n V\,, \quad \text{where } S^n V \overset{\text{def}}{=} V^{\otimes n} / (I_{\text{com}} \cap V^{\otimes n})\,.$$

EXERCISE 4.9. Show that for every basis $e_1, e_2, \dots, e_d \subset V$, the monomials $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ form a basis of SV over $\Bbbk$.

Thus, we get an isomorphism of algebras $SV \simeq \Bbbk[e_1, e_2, \dots, e_d]$. Under this isomorphism, $S^n V$ turns to the subspace of homogeneous polynomials of degree n.

EXERCISE 4.10. Deduce from the universal property of tensor multiplication that the map

$$V \times V \times \cdots \times V \to S^n V$$

provided by the multiplication in SV is the universal symmetric multilinear map. Convince yourself that SV is the *free commutative* $\Bbbk$*-algebra* spanned by V in the sense that for every commutative $\Bbbk$-algebra A and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : SV \to A$ such that $f = \widetilde{\varphi} \circ \iota$, where $\iota : V \hookrightarrow SV$ embeds V in SV as the space of linear homogeneous polynomials. Show that the latter embedding is uniquely characterized by the previous universal property up to a unique isomorphism commuting with ι.

4.3.2 The exterior[1] algebra of a vector space V is defined as the factor algebra $\Lambda V \overset{\text{def}}{=} \mathsf{T}V / I_{\text{alt}}$, where $I_{\text{alt}} \subset \mathsf{T}V$ is the two-sided ideal generated by all tensor squares $v \otimes v$, $v \in V$.

EXERCISE 4.11. Check that the space $I_{\text{alt}} \cap V^{\otimes 2}$ contains all sums $v \otimes w + w \otimes v$, $v, w \in V$, and is linearly generated over $\Bbbk$ by these sums if $\operatorname{char} \Bbbk \neq 2$.

The ideal I_{alt} also splits in the direct sum of homogeneous components

$$I_{\text{alt}} = \bigoplus_{n\geqslant 0} \left(I_{\text{alt}} \cap V^{\otimes n}\right)\,.$$

The degree n component $I_{\text{alt}} \cap V^{\otimes n}$ is spanned by decomposable tensors of the form

$$(\cdots \otimes v \otimes v \otimes \cdots)\,, \quad v \in V\,.$$

By the Exercise 4.11, all the sums $(\cdots \otimes v \otimes w \otimes \cdots) + (\cdots \otimes w \otimes v \otimes \cdots)$ belong to $I_{\text{alt}} \cap V^{\otimes n}$ as well and linearly generate it over $\Bbbk$ as soon $\operatorname{char} \Bbbk \neq 2$. The multiplication in ΛV is called the

[1] Also known as the *grassmannian algebra* or *free super-commutative algebra* of V.

exterior[1] *multiplication* and denoted by the wedge sign $\wedge$. Note that for any $u, w \in V$, the relations $u \wedge u = 0$ and $u \wedge w = -w \wedge u$ hold in $\Lambda^2 V$. Hence, under a permutation of factors, the exterior product of vectors is multiplied by the sign of permutation:

$$\forall\, g \in S_k \quad v_1 \wedge v_2 \wedge \cdots \wedge v_k = \operatorname{sgn}(g) \cdot v_{g_1} \wedge v_{g_2} \wedge \cdots \wedge v_{g_k}\,.$$

This property of a multiplication is known as the *super-commutativity*. Like the symmetric algebra, the exterior algebra is graded:

$$\Lambda V = \bigoplus_{n \geqslant 0} \Lambda^n V\,, \quad \text{where } \Lambda^n V \overset{\text{def}}{=} V^{\otimes n} / (I_{\text{alt}} \cap V^{\otimes n})\,.$$

EXERCISE 4.12. Deduce from the universal property of tensor multiplication that the map

$$V \times V \times \cdots \times V \to \Lambda^n V$$

provided by the exterior multiplication in ΛV is the universal alternating multilinear map. Convince yourself that ΛV is the *free super-commutative* $\Bbbk$*-algebra* spanned by V in the sense that for every super-commutative $\Bbbk$-algebra A and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : SV \to A$ such that $f = \widetilde{\varphi} \circ \iota$, where $\iota : V \hookrightarrow SV$ embeds V in ΛV as the subspace $\Lambda^1 V = V^{\otimes 1}$. Show that the latter embedding is uniquely characterized by the previous universal property up to a unique isomorphism commuting with ι.

PROPOSITION 4.1
For every basis $e_1, e_2, \dots, e_d$ in V the grassmannian monomials $e_I \overset{\text{def}}{=} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, numbered by strictly increasing multi-indexes $I = (i_1, i_2, \dots, i_n)$, $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$, form a basis of $\Lambda^n V$.

PROOF. Write U for the vector space of dimension $\binom{d}{n}$ with the basis formed by symbols ξ_I, where $I = (i_1, i_2, \dots, i_n)$ runs through all strictly increasing sequences of length n in $1, 2, \dots, d$. Consider the multilinear map $\alpha : V \times V \times \cdots \times V \to U$ that takes an arbitrary collection $e_{j_1}, e_{j_2}, \dots, e_{j_n}$ of the basis vectors from V to $\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n}) = \operatorname{sgn}(\sigma) \cdot \xi_I$, where $I = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(n)})$ is the strictly increasing permutation of the indexes $j_1, j_2, \dots, j_n$ and we put $\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n}) = 0$ when some of j_ν's coincide. For any alternating multilinear map $\varphi : V \times V \times \cdots \times V \to W$, there exists a unique linear operator $F : U \to W$ such that $\varphi = F \circ \alpha$: the action F on the basis of U has to be $F(\xi_{(i_1, i_2, \dots, i_n)}) = \varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n})$. Thus, α is the universal alternating multilinear map. Hence, there exists an isomorphism $U \simeq \Lambda^n V$ sending $\xi_I \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. □

COROLLARY 4.1
$\dim \Lambda^n V = \binom{d}{n}$, where $d = \dim V$. In particular, $\Lambda^n V = 0$ for $n > d$, and $\dim \Lambda V = 2^d$.

EXERCISE 4.13. Check that $\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha$ for any $\alpha \in \Lambda^a V$, $\beta \in \Lambda^b V$, and describe the centre[2] $Z(\Lambda V)$.

[1] Or *grassmannian*, or *super-commutative*
[2] That is, all elements commuting with every element of the algebra.

4.3.3 Grassmannian polynomials. It follows from the Proposition 4.1 that every choice of basis $e_1, e_2, \dots, e_d$ in a vector space V assigns the isomorphism of $\Bbbk$-algebras

$$\Lambda V \simeq k\langle e_1, e_2, \dots, e_d\rangle \,,$$

where $k\langle e_1, e_2, \dots, e_d\rangle$ stays for the algebra of *grassmannian polynomials*, i.e., polynomials with coefficients from $\Bbbk$ in the variables e_i satisfying the relations $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. When work with the grassmannian polynomials, we always write $I = (i_1, i_2, \dots, i_n)$ for a strictly increasing collection of indexes, $\hat{I} = (\hat{\imath}_1, \hat{\imath}_2, \dots, \hat{\imath}_{d-n}) = \{1, 2, \dots, d\} \smallsetminus I$ for the complementary strictly increasing collection, and $\#I \overset{\text{def}}{=} n$ for the *length* of I. The sum $|I| \overset{\text{def}}{=} \sum_\nu i_\nu$ is called the *weight* of I.

EXERCISE 4.14. Check that $e_I \wedge e_{\hat{I}} = (-1)^{|I|+\frac{1}{2}\#I(1+\#I)} \cdot e_1 \wedge e_2 \wedge \dots \wedge e_d$.

EXAMPLE 4.3 (LINEAR SUBSTITUTION OF VARIABLES)
Let the variables $e_1, e_2, \dots, e_n$ be linearly expressed through the variables $\xi_1, \xi_2, \dots, \xi_m$ as

$$e_i = \sum_j a_{ij}\,\xi_j \tag{4-20}$$

for some $n \times m$ matrix $A = \left(a_{ij}\right)$. Then the grassmannian monomials e_I are expressed through ξ_I as

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} = \Big(\sum_{j_1} a_{i_1 j_1}\,\xi_{j_1}\Big) \wedge \Big(\sum_{j_2} a_{i_2 j_2}\,\xi_{j_2}\Big) \wedge \dots \wedge \Big(\sum_{j_n} a_{i_n j_n}\,\xi_{j_n}\Big) =$$
$$= \sum_{1 \leqslant j_1 < j_2 < \dots < j_n \leqslant n} \; \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{i_1 j_{\sigma(1)}} a_{i_2 j_{\sigma(2)}} \cdots a_{i_n j_{\sigma(n)}}\, \xi_{j_1} \wedge \xi_{j_2} \wedge \dots \wedge \xi_{j_n} = \sum_J a_{IJ}\,\xi_J \,,$$

where J runs through increasing collections of length n and a_{IJ} denotes the $n \times n$ minor of A situated in the rows $i_1, i_2, \dots, i_n$ and columns $j_1, j_2, \dots, j_n$.

EXAMPLE 4.4 (MULTIROW COFACTOR EXPANSIONS OF DETERMINANT)
Let us perform the substitution (4-20) in the identity from the Exercise 4.14 using a square $d \times d$ matrix A. The left hand side of the identity turns to

$$\Big(\sum_{\substack{K:\\ \#K=\#I}} a_{IK}\,\xi_K\Big) \wedge \Big(\sum_{\substack{L:\\ \#L=(d-\#I)}} a_{\hat{I}L}\,\xi_L\Big) = (-1)^{\frac{1}{2}\#I(1+\#I)} \sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|} a_{IK} a_{\hat{I}\hat{K}}\, \xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_d \,.$$

The right hand side becomes $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|} \det(a_{ij}) \cdot \xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_d$. Thus, for every collection $I = (i_1, i_2, \dots, i_n)$ of rows in a square matrix $A = (a_{ij})$, the following relation holds

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} a_{\hat{I}\hat{K}} = \det(a_{ij}) \,, \tag{4-21}$$

where the summation goes over all $n \times n$ minors a_{IK} situated in the rows $(i_1, i_2, \dots, i_n)$.

If we replace $\hat{I}$ by another collection $\hat{J}$ complementary to the other $J \neq I$, then we get in the right hand side $e_I \wedge e_{\hat{J}} = 0$. Thus, for every $J \neq I$,

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} a_{\hat{J}\hat{K}} = 0 \,. \tag{4-22}$$

The identities (4-21) and (4-22) are known as the *Laplace relations*. They generalize the cofactor expansions of determinants. If we organize $n\times n$ minors of A and their complements in two $\binom{d}{n}\times\binom{d}{n}$ matrices $\mathcal{A}_n = \left(a_{IJ}\right)$ and $\mathcal{A}_n^\vee = \left(a_{IJ}^\vee\right)$, where[1] $a_{IJ}^\vee = (-1)^{|I|+|J|}a_{\hat{J}\hat{I}}$, then all the Laplace relations can be combined in the one matrix identity $\mathcal{A}_n \cdot \mathcal{A}_n^\vee = \det A \cdot E$.

EXERCISE 4.15. Write the Laplace relations for multicolumn cofactor expansions and prove that $\mathcal{A}_n^\vee \cdot \mathcal{A}_n = \det A \cdot E$ as well.

EXAMPLE 4.5 (REDUCTION OF GRASSMANNIAN QUADRATIC FORM)
Certainly, a grassmannian quadratic form can not be reduced to a «sum of squares» like in the Proposition 2.1 on p. 16. However, every homogeneous grassmannian polynomial of degree two over an arbitrary field $\Bbbk$ takes in appropriate coordinates the form

$$\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots + \xi_{2r-1} \wedge \xi_{2r}, \tag{4-23}$$

called *the Darboux normal form*. To achieve it for a given $\omega \in \Lambda^2 V$, we renumber the initial basis $e_1, e_2, \dots, e_n$ of V in such a way that $\omega = e_1 \wedge (\alpha_2 e_2 + \cdots + \alpha_n e_n) + e_2 \wedge (\beta_3 e_3 + \cdots + \beta_n e_n) +$ (terms without e_1, e_2), where $\alpha_2 \neq 0$. Then we pass to the new basis $\{e_1, \xi_2, e_3, \dots, e_n\}$ which has $\xi_2 = \alpha_2 e_2 + \cdots + \alpha_n e_n$. The substitution $e_2 = \left(\xi_2 - \beta_3 e_3 - \cdots - \beta_n e_n\right)/\alpha_2$ in ω leads to

$$\omega = e_1 \wedge \xi_2 + \xi_2 \wedge (\gamma_3 e_3 + \cdots + \gamma_n e_n) + (\text{terms without } \xi_2) = \\ = (e_1 - \gamma_3 e_3 - \cdots - \gamma_n e_n) \wedge \xi_2 + (\text{terms without } e_1,\, \xi_2)\,.$$

Now we pass to the basis $\{\xi_1, \xi_2, e_3, \dots, e_n\}$, where $\xi_1 = e_1 - \gamma_3 e_3 - \cdots - \gamma_n e_n$. In this basis,

$$\omega = \xi_1 \wedge \xi_2 + (\text{terms without } \xi_1,\, \xi_2)$$

and we can continue by induction.

CONVENTION 4.1. In the rest of §4 we assume on default that $\operatorname{char}(\Bbbk) = 0$.

4.4 Symmetric and alternating tensors. The symmetric group S_n acts on $V^{\otimes n}$ by permutations of factors in decomposable tensors: for $g \in S_n$, we put

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{g(1)} \otimes v_{g(2)} \otimes \cdots \otimes v_{g(n)}\,. \tag{4-24}$$

Since the right hand side is multilinear in $v_1, v_2, \dots, v_n$, this formula assigns the well defined linear map $g : V^{\otimes n} \to V^{\otimes n}$.

DEFINITION 4.3
A tensor $t \in V^{\otimes n}$ is called *symmetric*, if $g(t) = t$ for all $g \in S_n$. A tensor $t \in V^{\otimes n}$ is called *alternating*, if $g(t) = \operatorname{sgn}(g) \cdot t$ for all $g \in S_n$. We write $\operatorname{Sym}^n V = \{\, t \in V^{\otimes n} \mid \forall g \in S_n\ \sigma(t) = t\}$ and $\operatorname{Alt}^n V = \{\, t \in V^{\otimes n} \mid \forall g \in S_n\ g(t) = \operatorname{sgn}(g)\}$ for the space of symmetric and alternating tensors respectively. Note that both are the *sub*spaces in $V^{\otimes n}$, and they should not be confused with the *quotient* spaces $S^n V$, $\Lambda^n V$ of $V^{\otimes n}$.

[1] Note that I, J swap places.

4.4.1 Standard bases. For every basis $e_1, e_2, \dots, e_d$ in V, a basis of $\operatorname{Sym}^n V$ is formed by the *complete symmetric tensors*

$$e_{[m_1,m_2,\dots,m_d]} \overset{\text{def}}{=} \left(\begin{matrix}\text{the sum of all tensor monomials containing}\\ m_1 \text{ factors } e_1,\ m_2 \text{ factors } e_2,\ \dots,\ m_d \text{ factors } e_d\,,\end{matrix}\right) \tag{4-25}$$

because all the summands appear in the expansion of every symmetric tensor t with equal coefficients. The tensors (4-25) are indexed by the collections of non-negative integers $(m_1, m_2, \dots, m_d)$ such that $\sum_\nu m_\nu = n$.

EXERCISE 4.16. Make it sure that the sum (4-25) consists of $\frac{n!}{m_1!\,m_2!\cdots m_d!}$ terms.

Similarly, a basis of $\operatorname{Alt}^n V$ is formed by the *complete alternating tensors*

$$e_{\langle i_1,i_2,\dots,i_n\rangle} \overset{\text{def}}{=} \sum_{g\in S_n} \operatorname{sgn}(g)\cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}} \tag{4-26}$$

numbered by increasing sequences $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

4.5 Polarization of commutative polynomials. The quotient map $V^{\otimes n} \twoheadrightarrow S^n V$ sends every summand of (4-25) to the same commutative monomial $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. Thus, this map sends $e_{[m_1,m_2,\dots,m_d]}$ to $\frac{n!}{m_1!\,m_2!\cdots m_d!}\cdot e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. Over the ground field of zero characteristic, we conclude that for every n, the factorization through the commutativity relations assigns the isomorphism $\operatorname{Sym}^n V \simeq S^n V$. The inverse isomorphism is denoted by

$$\operatorname{pl} : S^n V \simeq \operatorname{Sym}^n V\,, \quad f \mapsto \widetilde{f}\,,$$

and called the *complete polarization* of polynomials. For the dual space V^*, the complete polarization map $\operatorname{pl} : S^n V^* \simeq \operatorname{Sym}^n V^*$ sends every monomial $f = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$ to the tensor $\widetilde{f} = \frac{m_1!\,m_2!\cdots m_d!}{n!}\cdot x_{[m_1,m_2,\dots,m_d]} \in \operatorname{Sym}^n V^*$, which can be viewed as the symmetric multilinear map $\widetilde{f} : V\times V\times \dots \times V \to \Bbbk$ acting on a collection of vectors $v_1, v_2, \dots, v_n \in V\times V \cdots \times V$ via the complete contraction with $v_1 \otimes v_2 \otimes \cdots \otimes v_n$.

EXERCISE 4.17. Verify that for every $v \in V$, the complete contraction of $v^{\otimes n}$ with

$$\frac{m_1!\,m_2!\,\cdots\,m_d!}{n!}\cdot x_{[m_1,m_2,\dots,m_d]}$$

is equal to the result of evaluation of monomial $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \in \Bbbk[x_1, x_2, \dots, x_n]$ on the coordinates of v.

We conclude that the polynomial function $f : \mathbb{A}(V) \to \Bbbk$ attached to a homogeneous polynomial $f \in S^n V$ in n° 1.1.2 on p. 3 is described in coordinate-free terms as $f(v) = \widetilde{f}(v, v, \dots, v)$, where $\widetilde{f} \in \operatorname{Sym}^n V^* \subset V^{*\otimes n}$ is the unique symmetric tensor mapped to f under factorization through the commutativity relations and considered as a symmetric multilinear map $V\times V\times \cdots \times V \to \Bbbk$. For $n = 2$, we get the polarization of quadratic forms considered in n° 2.1.1 on p. 16.

Since the value $\widetilde{f}(v_1, v_2, \dots, v_n)$ does not depend on the order of arguments, we write

$$\widetilde{f}\left(w_1^{k_1}, w_2^{k_2}, \dots, w_s^{k_s}\right)$$

when the collection $(v_1, v_2, \dots, v_n)$ consists of k_1 vectors w_1, k_2 vectors w_2, … , k_s vectors w_s.

EXERCISE 4.18. For any polynomial $f \in S^nV^*$ and vectors $v_1, v_2, \dots, v_k \in V$, show that

$$f\left(v_1 + v_2 + \dots + v_k\right) = \widetilde{f}\left(\left(v_1 + v_2 + \dots + v_k\right)^n\right) = \sum_{m_1 m_2 \dots m_k} \frac{n!}{m_1!\, m_2! \cdots m_k!} \cdot \widetilde{f}\left(v_1^{m_1}, v_2^{m_2}, \dots, v_k^{m_k}\right), \tag{4-27}$$

where the summation goes over all integer $m_1, m_2, \dots, m_k$ such that $m_1 + m_2 + \dots + m_k = n$ and $0 \leqslant m_\nu \leqslant n$ for all ν.

PROPOSITION 4.2

The complete polarization of a homogeneous polynomial $f \in S^nV^*$ on a vector space[1] V over a field of zero characteristic can be computed by the formula

$$n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \sum_{I \subsetneq \{1,\dots,n\}} (-1)^{\#I} f\Big(\sum_{i \notin I} v_i\Big), \tag{4-28}$$

where the left summation goes over all proper subsets $I \subsetneq \{1, 2, \dots, n\}$, including $I = \varnothing$, for which we put $\#\varnothing = 0$.

EXAMPLE 4.6

For homogeneous quadratic and cubic polynomials $q \in S^2V^*$, $f \in S^3V^*$, we get

$$2\widetilde{q}(u, w) = q(u + w) - q(u) - q(w),$$
$$6\,\widetilde{f}(u, v, w) = f(u + v + w) - f(u + v) - f(u + w) - f(v + w) + f(u) + f(v) + f(w).$$

PROOF OF THE PROPOSITION 4.2. In the expansion (4-27) for

$$f(v_1 + v_2 + \dots + v_n) = \widetilde{f}\left((v_1 + v_2 + \dots + v_n)^n\right),$$

there is just one term containing all the vectors $v_1, v_2, \dots, v_n$, namely $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n)$. For a proper subset $I \subsetneq \{1, 2, \dots, n\}$, every summand which contains no v_i with $i \in I$ appears in (4-27) with the same coefficient as in the expansion (4-27) written for $f(\sum_{i \notin I} v_i)$, because the latter is obtained from $f(v_1 + v_2 + \dots + v_n)$ by setting $v_i = 0$ for all $i \in I$. Removal of these summands via the standard combinatorial inclusion-exclusion procedure leads to the required formula

$$n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = f\Big(\sum_{\nu} v_\nu\Big) - \sum_{\{i\}} f\Big(\sum_{\nu \neq i} v_\nu\Big) + \sum_{\{i,j\}} f\Big(\sum_{\nu \neq i,j} v_\nu\Big) - \sum_{\{i,j,k\}} f\Big(\sum_{\nu \neq i,j,k} v_\nu\Big) + \cdots.$$

□

[1] Not necessary finite dimensional.

4.5.1 Duality. For a vector space V of finite dimensuon over a field of zero characteristic, the complete contraction between $V^{\otimes m}$ and $V^{*\otimes m}$ provides the spaces S^mV and S^mV^* with the perfect pairing that couples polynomials $f \in S^nV$ and $g \in S^nV^*$ to the complete contraction of their complete polarizations $\widetilde{f} \in V^{\otimes m}$ and $\widetilde{g} \in V^{*\otimes m}$.

EXERCISE 4.19. For a pair of dual bases $e_1, e_2, \dots, e_d \in V$, $x_1, x_2, \dots, x_d \in V^*$, verify that all the non-zero couplings between the basis monomials are exhausted by

$$\langle\, e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \,,\, x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \,\rangle = \frac{m_1!\, m_2! \cdots m_d!}{n!}\,. \qquad (4\text{-}29)$$

Note that the monomials constructed from the dual basis vectors become the dual bases of the polynomial rings only after rescaling by appropriate combinatorial factors.

4.5.2 Derivative of a polynomial along a vector. Associated with every vector $v \in V$ is the linear map $i_v : V^{*\otimes n} \to V^{*\otimes(n-1)}$, $\varphi \mapsto i_v\varphi$, provided by the inner multiplication[1] of n-linear forms on V by v, which takes an n-linear form $\varphi \in V^{*\otimes n}$ to the $(n-1)$-linear form

$$i_v\varphi(v_1, v_2, \dots, v_{n-1}) = \varphi(v, v_1, v_2, \dots, v_{n-1})\,.$$

Composing this map with preceded complete polarization$S^nV^* \xrightarrow{\sim} \mathrm{Sym}^n V^* \subset V^{*\otimes n}$ and subsequent factorization $\sigma : V^{*\otimes(n-1)} \twoheadrightarrow S^{n-1}V^*$ through the commutativity relations[2], assigns the linear map

$$\mathrm{pl}_v : S^nV^* \to S^{n-1}V^*\,, \quad f(x) \mapsto \mathrm{pl}_v f(x) \stackrel{\text{def}}{=} \widetilde{f}(v,\, x,\, x,\, \dots\,,\, x)\,, \qquad (4\text{-}30)$$

which depends linearly on $v \in V$. This map fits in the commutative diagram

$$\begin{array}{ccc} V^{*\otimes n} \supset \mathrm{Sym}^n V^* & \xrightarrow{\;i_v\;} & V^{*\otimes(n-1)} \\ \mathrm{pl}\,\big\uparrow\wr & & \big\downarrow\sigma \\ S^nV^* & \xrightarrow{\;\mathrm{pl}_v\;} & S^{n-1}V^*\,. \end{array} \qquad (4\text{-}31)$$

The polynomial $\mathrm{pl}_v f(x)\widetilde{f}(v, x, \dots\, x) \in S^{n-1}(V^*)$ is called the *polar* of v with respect to f. For $n = 2$, the polar of a vector v with respect to a quadratic worm $f \in S^2V^*$ is the linear form $w \mapsto \widetilde{f}(v, w)$ considered[3] in n° 2.2.1 on p. 18.

In terms of dual bases $e_1, e_2, \dots, e_d \in V$, $x_1, x_2, \dots, x_d \in V^*$, the contraction of the first tensor factor in $V^{*\otimes n}$ with the basis vector $e_i \in V$ maps the complete symmetric tensor $x_{[m_1, m_2, \dots, m_n]}$ either to the complete symmetric tensor containing the $(m_i - 1)$ factors x_i or to zero for $m_i = 0$. Hence, $pl_{e_i} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} = \frac{m_i}{n} x_1^{m_1} \dots x_{i-1}^{m_{i-1}} x_i^{m_i-1} x_{i+1}^{m_{i+1}} \dots x_d^{m_d} = \frac{1}{n} \frac{\partial}{\partial x_i} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$. Since $\mathrm{pl}_v f$ is linear in both v, f, we conclude that for every $v = \sum \alpha_i e_i$, the polar polynomial of v with respect to f is nothing but the *derivative* of the polynomial f along the vector v divided by $\deg f$, i.e.,

$$\mathrm{pl}_v f = \frac{1}{\deg(f)}\, \partial_v f = \frac{1}{\deg(f)} \sum_{i=1}^{d} \alpha_i \frac{\partial f}{\partial x_i}\,.$$

[1]See the Example 4.2 on p. 42.

[2]Which is the linear map corresponding to the commutative multiplication of covectors from formula (4-16) on p. 43 by the universal property of tensor product.

[3]Recall that the zero set of this form in $\mathbb{P}(V)$ is the hyperplane intersecting the quadric $V(f) \subset \mathbb{P}(V)$ along its apparent contour viewed from v.

Note that this forces the right hand side to be independent on the choice of dual bases in V and V^*. It follows from the definition of polar map that the derivatives along vectors commute, $\partial_u\partial_w = \partial_w\partial_u$, and for all $u, w \in V$, $f \in S^nV^*$, $0 \leqslant m \leqslant n$, the following relation holds:

$$m!\,\frac{\partial^m f}{\partial u^m}(w) = n!\,\widetilde{f}(u^m, w^n) = (n-m)!\,\frac{\partial^{n-m} f}{\partial w^{n-m}}(u)\,, \tag{4-32}$$

EXERCISE 4.20. Prove *the Leibniz rule* $\partial_v(fg) = \partial_v(f)\cdot g + f\cdot\partial_v(g)$ and show that

$$\widetilde{f}(v_1, v_2, \dots, v_n) = \frac{1}{n!}\partial_{v_1}\partial_{v_2}\,\dots\,\partial_{v_n} f\,.$$

EXAMPLE 4.7 (TAYLOR'S EXPANSION)
For $k = 2$, the expansion (4-27) from the Exercise 4.18 turns to the identity

$$f(u+w) = \widetilde{f}(u+w,\, u+w,\, \dots,\, u+w) = \sum_{m=0}^{n}\binom{n}{m}\cdot\widetilde{f}(u^m, w^{n-m}),$$

where $n = \deg f$. It holds for any polynomial $f \in S^nV^*$ and all vectors $u, w \in V$. The relations (4-32) allow us to rewrite this identity as the *Taylor expansion* for f at u:

$$f(u+w) = \sum_{m=0}^{\deg f}\frac{1}{m!}\,\partial_w^m f(u), \tag{4-33}$$

which is an exact equality in the polynomial ring SV^*.

4.5.3 Polars and tangents. Given a hypersurface $S = V(f) \subset \mathbb{P}(V)$ of degree n and a line $\ell = (pq) \subset \mathbb{P}(V)$, the intersection $\ell\cap S$ consists of all points $\lambda p+\mu q$ such that $(\lambda : \mu) \in \mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ is a root of the homogeneous polynomial $f_{pq}(\lambda,\mu) \stackrel{\text{def}}{=} f(\lambda p+\mu q) \in \Bbbk[\lambda,\mu]$. Over an algebraically closed field $\Bbbk$, this polynomial is either zero or a product of n non-zero homogeneous linear forms in λ, μ, possibly coinciding:

$$f(\lambda,\mu) = \prod_i(\alpha_i''\lambda - \alpha_i'\mu)^{s_i} = \prod_i \det{}^{s_i}\begin{pmatrix}\lambda & \alpha_i'\\ \mu & \alpha_i''\end{pmatrix}\,, \tag{4-34}$$

where $a_i = (\alpha_i' : \alpha_i'')$ are some mutually distinct points on $\mathbb{P}_1$ and $\sum_i s_i = n$. If $f_{pq} = 0$, then $\ell \subset S$. If $f_{pq} \neq 0$, then the intersection $\ell\cap S$ consists of the points $a_i = \alpha_i' p + \alpha_i'' q$. The exponent s_i of the linear form $\alpha_i''\mu - \alpha_i'\lambda$ in the factorization (4-34) is called the *intersection multiplicity* of the hypersurface S with the line ℓ at the point a_i, and is denoted by $(S,\ell)_{a_i}$. If $(S,\ell)_{a_i} = 1$, the intersection point a_i is called *simple* or *transversal*. Otherwise, the intersection of ℓ and S at a_i is called a *multiple*. The total number of intersections counted with their multiplicities equals the degree of S.

A line $\ell = (pq)$ passing through $p \in S$ is called *tangent* to S at p if either $\ell \subset S$ or $(S,\ell)_p \geqslant 2$. In other words, the line ℓ is tangent to S at p if the polynomial $f(p+tq) \in \Bbbk[t]$ either is the zero polynomial or has a multiple root at zero. The Taylor expansion[1] for $f(p+tq)$ at p starts with

$$f(p+tq) = t\binom{d}{1}\widetilde{f}(p^{n-1}, q) + t^2\binom{d}{2}\widetilde{f}(p^{n-2}, q^2) + \cdots\,.$$

[1]See 4-33 on p. 51.

Therefore the line $\ell = (pq)$ is tangent to S at p if and only if $\widetilde{f}(p^{n-1}, q) = 0$. This is the straightforward generalization of the Proposition 2.2 on p. 17.

If $f(p^{n-1}, x)$ does not vanish identically as a linear form in x, the point p is called a *smooth point* of S. The hypersurface $S \subset \mathbb{P}(V)$ is called *smooth* if every point $p \in S$ is smooth. For a smooth $p \in S$ the linear equation $F(p^{n-1}, x) = 0$ on $x \in V$ defines a hyperplane in $\mathbb{P}(V)$ filled by the lines (pq) tangent to S at p. This hyperplane is called the *tangent space* to S at p and denoted by $T_p = \{x \in \mathbb{P}(V) \mid \widetilde{f}(p^{n-1}, x) = 0\}$.

If $f(p^{n-1}, x)$ is the zero linear form in x, the hypersurface S is called *singular* at p, and the point p is called a *singular point* of S. Since the coefficients of polynomial $\widetilde{f}(p^{n-1}, x) = \partial_x f(p)$, considered as a linear form in x, are equal to the partial derivatives of f evaluated at the point p by (4-32), the singularity of $p \in S = V(f)$ is expressed by the equations

$$\frac{\partial f}{\partial x_i}(p) = 0 \quad \text{for all } i\,,$$

in which case any line ℓ passing through p has $(S, \ell)_p \geqslant 2$, i.e., is tangent to S at p. Thus, the tangent lines to S at a singular point of S fill the whole ambient space $\mathbb{P}(V)$.

If q is either a smooth point on S or a point outside S, then the polar polynomial

$$\mathrm{pl}_q f(x) = \widetilde{f}(q, x^{n-1})$$

does not vanish identically as a homogeneous polynomial of degree $n-1$ in x, because otherwise, all partial derivatives of $\mathrm{pl}_q f(x) = \widetilde{f}(q, x^{n-1})$ in x would also vanish, and in particular,

$$\widetilde{f}(q^{n-1}, x) = \frac{\partial^{n-2}}{\partial q^{n-2}} \mathrm{pl}_q f(x) = 0$$

identically in x, meaning that q is a singular point of S, in contradiction with our choice of q. The zero set of the polar polynomial $\mathrm{pl}_q f \in S^{n-1}V^*$ is denoted by

$$\mathrm{pl}_q S \overset{\text{def}}{=} V(\mathrm{pl}_q f) = \{x \in \mathbb{P}(V) \mid \widetilde{f}(q, x^{n-1}) = 0\} \tag{4-35}$$

and called the *polar hypersurface* of the point q with respect to S. If S is a quadric, then $\mathrm{pl}_q S$ is exactly the polar hyperplane of q considered in n° 2.3.1 on p. 19. As in the Corollary 2.2 on p. 17, for a hypersurface S of arbitrary degree, the intersection $S \cap \mathrm{pl}_q S$ coincides with the *apparent contour* of S viewed from the point q, that is, with the locus of all points $p \in S$ such that the line (pq) is tangent to S at p.

More generally, for an arbitrary point $q \in \mathbb{P}(V)$ the locus of points

$$\mathrm{pl}_q^{n-r} S \overset{\text{def}}{=} \left\{ x \in \mathbb{P}(V) \mid \widetilde{f}(q^{n-r}, x^r) = 0 \right\}$$

is called the *rth degree polar* of the point q with respect to S or *the rth degree polar* of S at q for $q \in S$. If the polynomial $\widetilde{f}(q^{n-r}, x^r)$ vanishes identically in x, we say that the rth degree polar is *degenerate*. Otherwise, the rth degree polar is a projective hypersurface of degree r. The linear[1] polar of S at a smooth point $q \in S$ is simply the tangent hyperplane to S at q: $\mathrm{pl}_q^{n-1} S = T_q S$. The quadratic polar $\mathrm{pl}_q^{n-2} S$ is the quadric passing through q and having the same tangent hyperplane at q as S. The cubic polar $\mathrm{pl}_q^{n-3} S$ is the cubic hypersurface passing through q and having the same quadratic polar at q as S, etc. The rth degree polar $\mathrm{pl}_q^{n-2} S$ at a smooth point $q \in S$ passes through q and has $\mathrm{pl}_q^{r-k} \mathrm{pl}_q^{n-r} S = \mathrm{pl}_q^{n-k} S$ for all $1 \leqslant k \leqslant r-1$, because

$$\mathrm{pl}_q^{r-k} \mathrm{pl}_q^{n-r} f(x) = \widetilde{\mathrm{pl}_q^{n-r} f}(q^{r-k}, x^k) = \widetilde{f}(q^{n-r}, q^{r-k}, x^k) = \widetilde{f}(q^{n-k}, x^k) = \mathrm{pl}_q^{n-k} f(x)\,.$$

[1] That is, of the first degree.

4.5.4 Linear support of a homogeneous polynomial. For a polynomial $f \in S^n V^*$, we write $\operatorname{Supp} f$ for the minimal[1] vector subspace $W \subset V^*$ such that $f \in S^n W$, and call it the *linear support* of f. Over a field of zero characteristic, $\operatorname{Supp} f = \operatorname{Supp} \widetilde{f}$, where $\widetilde{f} \in \operatorname{Sym}^n V^* \subset V^{*\otimes n}$ is the complete polarization of f. By the Theorem 4.1, $\operatorname{Supp} \widetilde{f}$ is linearly generated by the images of the $(n-1)$-tuple contraction maps

$$c_t^J : V^{\otimes(n-1)} \to V^*, \quad t \mapsto c^{1,\,2,\,\dots\,,\,(n-1)}_{j_1,j_2,\dots,j_{n-1}}(t \otimes \widetilde{f}),$$

coupling all the $(n-1)$ factors of $V^{\otimes(n-1)}$ with some $n-1$ factors of $\tilde{t} \in V^{*\otimes n}$ in order indicated by the sequence $J = (j_1, j_2, \dots, j_{n-1})$. For the symmetric tensor $\widetilde{f}$, such a contraction does not depend on J and maps every decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1}$ to the linear form on V proportional to the derivative $\partial_{v_1}\partial_{v_2} \dots \partial_{v_{n-1}} f \in V^*$. Thus, $\operatorname{Supp}(f)$ is linearly generated by all $(n-1)$-tuple partial derivatives

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}}\frac{\partial^{m_2}}{\partial x_2^{m_2}} \cdots \frac{\partial^{m_d}}{\partial x_d^{m_d}} f(x), \quad \text{where } \sum m_\nu = n-1\,. \tag{4-36}$$

The coefficient of x_i in the linear form (4-36) depends only on the coefficients of monomial

$$x_1^{m_1} \dots x_{i-1}^{m_{i-1}} x_i^{m_i+1} x_{i+1}^{m_{i+1}} \dots x_d^{m_d}$$

in f. If we write the polynomial f as

$$f = \sum_{\nu_1+\cdots+\nu_d=n} \frac{n!}{\nu_1!\,\nu_2! \cdots \nu_d!}\, a_{\nu_1\nu_2\dots\nu_d}\, x_1^{\nu_1} x_2^{\nu_2} \cdots x_d^{\nu_d}\,, \tag{4-37}$$

the linear form (4-36) turns to

$$n! \cdot \sum_{i=1}^{d} a_{m_1\dots m_{i-1}(m_i+1)m_{i+1}\dots m_d}\, x_i\,. \tag{4-38}$$

Totally, we get $\binom{n+d-2}{d-1}$ such the linear forms staying in bijection with the nonnegative integer solutions $m_1, m_2, \dots, m_d$ of the equation $m_1 + m_2 + \cdots + m_d = n-1$.

PROPOSITION 4.3
Let $\Bbbk$ be a field of zero characteristic, V a finite dimensional vector space over $\Bbbk$, and $f \in S^n V^*$ a polynomial written in the form (4-37) in some basis of V^*. If $f = \varphi^n$ for some linear form $\varphi \in V^*$, then the $d\times\binom{n+d-2}{d-1}$ matrix built from the coefficients of linear forms (4-38) has rank 1. In this case, there are at most n linear forms $\varphi \in V^*$ such that $\varphi^n = f$, and they differ from one another by multiplications by the nth roots of unity laying in $\Bbbk$. For algebraically closed field $\Bbbk$, the converse is also true: if all the linear forms (4-38) are proportional, then $f = \varphi^n$ for some linear form φ proportional to the forms (4-38).

PROOF. The equality $f = \varphi^n$ means that $\operatorname{Supp}(f) \subset V^*$ is the 1-dimensional subspace spanned by φ. In this case, all linear forms (4-38) are proportional to φ. Such a form $\psi = \lambda\varphi$ has $\psi^n = f$ if and only if $\lambda^n = 1$ in $\Bbbk$. Conversely, let all the linear forms (4-38) be proportional, and $\psi \neq 0$ be one of them. Then, $\operatorname{Supp}(f) = \Bbbk \cdot \psi$ is the 1-dimensional subspace spanned by ψ. Hence, $f = \lambda\psi^n$ for some $\lambda \in \Bbbk$, and therefore, $f = \varphi^n$ for[2] $\varphi = \sqrt[n]{\lambda} \cdot \psi$. □

[1] With respect to inclusions.

[2] Here we use that $\Bbbk$ is algebraically closed.

4.5.5 The Veronese varieties $V(n,k)$. The *Veronese map*

$$v_{k,n}: \mathbb{P}(V^*) \hookrightarrow \mathbb{P}(S^nV^*), \quad \psi \mapsto \psi^n, \tag{4-39}$$

for $\dim V = k+1$ embeds $\mathbb{P}_k$ into $\mathbb{P}_N$, where $N = \binom{n+k}{k} - 1$. The image of map (4-39) is called the *Veronese variety* and denoted by $V(k,n) \subset \mathbb{P}(S^nV^*)$. It consists of perfect n th powers φ^n of linear forms $\varphi \in V^*$ considered up to proportionality. It follows from the Proposition 4.3 that $V(n,k)$ is indeed an algebraic projective variety described by a system of quadratic equations asserting the vanishing of all 2×2-minors in $d \times \binom{n+d-2}{d-1}$ matrix formed by the coefficients of the linear forms (4-38). For example, a homogeneous polynomial in two variables $f(x_0,x_1) = \sum_{k=0}^n a_k \binom{n}{k} x_0^{n-k}x_1^k$ has

$$\frac{\partial^{n-1}f}{\partial x_0^{n-i-1}\partial x_1^i} = n!\cdot(a_i x_0 + a_{i+1}x_1).$$

Hence, the image of the Veronese embedding $v_{1,n}: \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ is described by the condition

$$\operatorname{rk}\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1,$$

which agrees with the Example 1.4 on p. 11 and is equivalent to a system of quadratic equations

$$\det\begin{pmatrix} a_i & a_j \\ a_{i+1} & a_{j+1} \end{pmatrix} = 0$$

on the coefficients a_i of the polynomial f. A polynomial f satisfies these equations if and only if $f = \varphi^n$ for some linear form $\varphi = \alpha_0x_0 + \alpha_1x_1$, and in this case $(\alpha_0 : \alpha_1) = (a_i : a_{i+1})$ for all i.

4.6 Polarization of grassmannian polynomials. The quotient map $V^{\otimes n} \twoheadrightarrow \Lambda^nV$ sends every summand of the basis alternating tensor (4-26)

$$e_{\langle i_1,i_2,\dots,i_n\rangle} \stackrel{\text{def}}{=} \sum_{g\in S_n} \operatorname{sgn}(g)\cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \dots \otimes e_{i_{g(n)}}$$

to the same grassmannian monomial $e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}$. Thus, this map sends $e_{\langle i_1,i_2,\dots,i_n\rangle}$ to $n!\, e_I$, and therefore, over a field of zero characteristic, the factorization through the alternating relations assigns the isomorphism $\operatorname{Alt}^n V \xrightarrow{\sim} \Lambda^nV$. By analogy with the usual commutative polynomials, the inverse isomorphism is denoted by $\operatorname{pl}: \Lambda^nV \xrightarrow{\sim} \operatorname{Alt}^n V$, $\omega \mapsto \widetilde{\omega}$, and called the *complete polarization* of grassmannian polynomials.

4.6.1 Duality. For a finite dimensional vector space V over a field of zero characteristic, there is the perfect pairing between the spaces Λ^nV and Λ^nV^* coupling $\tau \in \Lambda^nV$ and $\omega \in \Lambda^nV^*$ to the complete contraction of their complete polarizations $\widetilde{\tau} \in V^{\otimes n}$ and $\widetilde{\omega} \in V^{*\otimes n}$.

EXERCISE 4.21. Convince yourself that the non zero couplings between the basis monomials $e_I \in \Lambda^nV$ and $x_J \in \Lambda^nV^*$ are exhausted by $\langle e_I, x_I\rangle = 1/n!$.

4.6.2 Partial derivatives in the exterior algebra. Given a covector $\psi \in V^*$, we write

$$\operatorname{pl}_\psi: \Lambda^nV \to \Lambda^{n-1}V$$

for the composition of inner multiplication $i_\psi: V^{\otimes n} \to V^{\otimes(n-1)}$ by ψ with preceding complete polarization $\operatorname{pl}: \Lambda^nV \xrightarrow{\sim} \operatorname{Alt}^n V$ and subsequent factorization $\alpha: V^{\otimes(n-1)} \twoheadrightarrow \Lambda^{n-1}V$ through the alternating relations[1]. Thus, pl_v fits in the commutative diagram

$$\begin{array}{ccc} V^{*\otimes n} \supset \mathrm{Skew}^n V^* & \xrightarrow{i_\psi} & V^{*\otimes(n-1)} \\ \mathrm{pl}\uparrow\wr & & \downarrow\alpha \\ \Lambda^n V^* & \xrightarrow{\mathrm{pl}_\psi} & \Lambda^{n-1}V^* \end{array} \tag{4-40}$$

similar to the diagram from formula (4-31) on p. 50. By analogy with n° 4.5.2, the polynomial

$$\partial_\psi \omega \overset{\text{def}}{=} \deg \omega \cdot \mathrm{pl}_\psi \omega$$

is called the *derivative* of homogeneous grassmannian polynomial $\omega \in \Lambda^n V$ in direction of covector $\psi \in V^*$. Since $\mathrm{pl}_\psi \omega$ is linear in ψ, the derivation along $\psi = \sum \alpha_i x_i$ splits as $\partial_\psi = \sum \alpha_i \partial_{x_i}$. If ω does not depend on e_i, then $\partial_{x_i}\omega = 0$. Therefore, a nonzero contribution to $\partial_\psi e_I$ is given only by the derivations ∂_{x_i} for $i \in I$.

Exercise 4.22. Check that $\partial_{x_{i_1}} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_{i_2} \wedge e_{i_3} \wedge \ldots \wedge e_{i_n}$ for every collection of indexes $i_1, i_2, \ldots, i_n$, not necessary increasing.

It follows from the Exercise 4.22 that

$$\begin{aligned} \partial_{x_{i_k}} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &= \partial_{x_{i_k}} (-1)^{k-1} e_{i_k} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \ldots e_{i_n} \\ &= (-1)^{k-1} \partial_{x_{i_k}} e_{i_k} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \ldots e_{i_n} \\ &= (-1)^{k-1} e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \ldots e_{i_n} . \end{aligned}$$

In other words, the derivation of a monomial along the basis covector dual to the kth variable from the left in the monomial behaves as $(-1)^{k-1}\partial/\partial e_{i_k}$, where the *grassmannian partial derivative* $\partial/\partial e_i$ takes e_i to 1 and annihilates all e_j with $j \neq i$, exactly as in the symmetric case. However, the sign $(-1)^k$ in the previous formula forces the grassmannian partial derivatives to satisfy *the grassmannian Leibniz rule*, which differs from the usual one by an extra sign.

Exercise 4.23 (the grassmannian Leibniz rule). For any homogeneous grassmannian polynomials $\omega, \tau \in \Lambda V$ and a covector $\psi \in V$, prove that

$$\partial_\psi(\omega \wedge \tau) = \partial_\psi(\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \partial_\psi(\tau) . \tag{4-41}$$

Since the grassmannian polynomials are linear in each variable, $\partial_\psi^2 \omega = 0$ for all $\psi \in V$, $\omega \in \Lambda V$. The relation $\partial_\psi^2 = 0$ forces the grassmannian derivatives to be super-commutative, that is,

$$\forall\, \psi, \xi \in V^* \quad \partial_\psi \partial_\xi = -\partial_\xi \partial_\psi .$$

4.6.3 Linear support of a homogeneous grassmannian polynomial. The *linear support* $\mathrm{Supp}\,\omega$ of a homogeneous grassmannian polynomial ω of degree n is defined to be the minimal[2] vector subspace $W \subset V$ such that $\omega \in \Lambda^n W$. It coincides with the linear support of the complete polarization $\widetilde{\omega} \in \mathrm{Skew}^n V$, and is linearly generated by all $(n-1)$-tuple partial derivatives[3]

$$\partial_J \omega \overset{\text{def}}{=} \partial_{x_{j_1}} \partial_{x_{j_2}} \ldots \partial_{x_{j_{n-1}}} \omega = \frac{\partial}{\partial e_{j_1}} \frac{\partial}{\partial e_{j_2}} \cdots \frac{\partial}{\partial e_{j_{n-1}}} \omega ,$$

[1] Which is the linear map corresponding to the alternating multiplication of covectors from formula (4-17) on p. 43 by the universal property of tensor product.

[2] With respect to inclusions.

[3] Compare with n° 4.5.4 on p. 53.

where $J = j_1 j_2 \dots j_{n-1}$ runs through all sequences of $n-1$ different indexes taken from the set $\{1, 2, \dots, d\}$, $d = \dim V$. Up to a sign, the order of indexes in J is not essential, and we will not assume the indexes to be increasing, because this simplifies the notations in what follows.

Let us expand ω as a sum of basis monomials

$$\omega = \sum_I a_I e_I = \sum_{i_1 i_2 \dots i_n} \alpha_{i_1 i_2 \dots i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}, \tag{4-42}$$

where $I = i_1 i_2 \dots i_n$ also runs through the n-tuples of different but non necessary increasing indexes, and the coefficients $\alpha_{i_1 i_2 \dots i_n} \in \Bbbk$ are alternating in $i_1 i_2 \dots i_n$. Nonzero contributions to $\partial_J \omega$ are given only by the monomials $a_I e_I$ with $I \supset J$. Therefore, up to a common sign,

$$\partial_J \omega = \pm \sum_{i \notin J} \alpha_{j_1 j_2 \dots j_{n-1} i} \, e_i \,. \tag{4-43}$$

Proposition 4.4

The following conditions on a grassmannian polynomial $\omega \in \Lambda^n V$ written in the form (4-42) are equivalent:

1) $\omega = u_1 \wedge u_2 \wedge \cdots \wedge u_n$ for some $u_1, u_2, \dots, u_n \in V$

2) $u \wedge \omega = 0$ for all $u \in \operatorname{Supp}(\omega)$

3) for any two collections $i_1 i_2 \dots i_{m+1}$ and $j_1 j_2 \dots j_{m-1}$ consisting of $n+1$ and $n-1$ different indexes, the following *Plücker relation* holds

$$\sum_{\nu=1}^{m+1} (-1)^{\nu-1} a_{j_1 \dots j_{m-1} i_\nu} a_{i_1 \dots \hat{i}_\nu \dots i_{m+1}} = 0\,, \tag{4-44}$$

where the hat in $a_{i_1 \dots \hat{i}_\nu \dots i_{m+1}}$ means that the index i_ν should be removed.

Proof. Condition (1) holds if and only if ω belongs to the top homogeneous component of its linear span, $\omega \in \Lambda^{\dim \operatorname{Supp}(\omega)} \operatorname{Supp}(\omega)$. Condition (2) means the same because of the following exercise.

Exercise 4.24. Show that $\omega \in \Lambda U$ is homogeneous of degree $\dim U$ if and only if $u \wedge \omega = 0$ for $u \in U$.

The Plücker relation (4-44) asserts the vanishing of the coefficient of $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{m+1}}$ in the product $\left(\partial_{j_1 \dots j_{m-1}} \omega\right) \wedge \omega$. In other words, (4-44) is the coordinate form of condition (2) written for vector $u = \partial_{j_1 \dots j_{m-1}} \omega$ from the formula (4-43). Since these vectors linearly generate the subspace $\operatorname{Supp}(\omega)$, the whole set of the Plücker relations is equivalent to the condition (2). □

Example 4.8 (the Plücker quadric)

Let $n = 2$, $\dim V = 4$, and e_1, e_2, e_3, e_4 be a basis of V. Then the expansion (4-42) for $\omega \in \Lambda^2 V$ looks like $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$, where the coefficients a_{ij} form the alternating 4×4 matrix. The Plücker relation corresponding to $(i_1, i_2, i_3) = (2, 3, 4)$ and $j_1 = 1$ is

$$a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0\,. \tag{4-45}$$

All other choices of (i_1, i_2, i_3) and $j_1 \notin \{i_1, i_2, i_3\}$ lead to exactly the same relation.

Exercise 4.25. Check this.

For $j_1 \in \{i_1, i_2, i_3\}$ we get the trivial equality $0 = 0$. Thus, for $\dim V = 4$, the set of decomposable grassmannian quadratic forms $\omega \in \Lambda^2 V$ is described by just one quadratic equation (5-2).

EXERCISE 4.26. Convince yourself that the equation (5-2) on $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$ is equivalent to the condition $\omega \wedge \omega = 0$.

4.6.4 The Grassmannian varieties and Plücker embeddins. For a vector space V of dimension d, the set of all vector subspaces $U \subset V$ of dimension m is denoted by $\mathrm{Gr}(m, V)$ and called the *grassmannian*. When the origin of V is not essential or $V = \Bbbk^d$, we write $\mathrm{Gr}(m, d)$ instead of $\mathrm{Gr}(m, V)$. Thus, $\mathrm{Gr}(1, V) = \mathbb{P}(V)$, $\mathrm{Gr}(\dim V - 1, V) = \mathbb{P}(V^*)$. The grassmannian $\mathrm{Gr}(m, V)$ is embedded into the projective space $\mathbb{P}\mathbb{P}(\Lambda^m V)$ by means of the *Plücker map*

$$p_m : \mathrm{Gr}(m, V) \to \mathbb{P}(\Lambda^m V), \quad U \mapsto \Lambda^m U \subset \Lambda^m V \tag{4-46}$$

sending every subspace $U \subset V$ of dimension m to its highest exterior power $\Lambda^m U$, which is a subspace of dimension 1 in $\Lambda^m V$. If U is spanned by vectors $u_1, u_2, \dots, u_m$, then up to proportionality, $p_m(U) = u_1 \wedge u_2 \wedge \cdots \wedge u_m$.

EXERCISE 4.27. Check that the Plücker map is injective.

The image of map (4-46) consists of all grassmannian polynomials $\omega \in \Lambda^m V$ completely factorisable into a product of m vectors. Such polynomials are called *decomposable*. By the Proposition 4.4 they form a projective algebraic variety described by the system of quadratic equations (4-44) on the coefficients of expansion (4-42).

REMARK 4.1. From the algebraic viewpoint, the grassmannian variety $\mathrm{Gr}(k, m) \subset \mathbb{P}(\Lambda^m V)$ is a super-commutative version of the Veronese variety $V(k, m) \subset \mathbb{P}(S^m V)$. Both consist of most degenerated non-zero homogeneous polynomials of degree m in the sense that the linear support of polynomial has the minimal possible dimension which equals 1 for a commutative polynomial, and equals m for a grassmannian polynomial of degree m.

EXAMPLE 4.9 (THE GRASSMANNIANS $\mathrm{Gr}(2, V)$)
The Plücker embedding identifies the grassmannian $\mathrm{Gr}(2, V)$ with the set of decomposable grassmannian quadratic forms $\omega \in \Lambda^2 V$, that is, $\omega = u \wedge w$ for some $u, w \in V$. Note that every such ω has $\omega \wedge \omega = u \wedge w \wedge u \wedge w = 0$. For an arbitrary $\omega \in \Lambda^2 V$, there exists a basis $\xi_1, \xi_2, \dots, \xi_d$ in V such that[1] $\omega = \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots$. If this sum contains more than one term, then the monomial $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$ appears in $\omega \wedge \omega$ with the coefficient 2 and therefore, $\omega \wedge \omega \neq 0$. Thus, such ω is not decomposable. We conclude that $\omega \in \Lambda^2 V$ is decomposable if and only if $\omega \wedge \omega = 0$.

For $\dim V = 4$, the squares of forms $\omega \in \Lambda^2 V$ lie in the space $\Lambda^4 V$ of dimension 1. In this case, the condition $\omega \wedge \omega = 0$ for $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$ is expressed by just one quadratic equation

$$a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0 \,, \tag{4-47}$$

which agrees with the equation (5-2) from the Example 4.8 on p. 56. We conclude that the Plücker embedding identifies the grassmannian $\mathrm{Gr}(2, 4) = \mathrm{Gr}(2, V)$ with the quadric (4-47) in $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. This quadric is called the *Plücker quadric*.

EXAMPLE 4.10 (THE SEGRE VARIETIES REVISITED[2])
Let $W = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ be a direct sum of finite dimensional vector spaces V_i. For every collection of non-negative integers $m_1, m_2, \dots, m_n$ such that $m_i \leqslant \dim V_i$, put $k = \sum_\nu m_\nu$ and

[1] See the Example 4.5 on p. 47.
[2] See n° 4.1.2 on p. 39.

denote by $W_{m_1,m_2,\dots,m_n} \subset \Lambda^k W$ the linear span of all products $w_1 \wedge w_2 \wedge \cdots \wedge w_k$ formed by m_1 vectors taken from V_1, m_2 vectors taken from V_2, etc.

EXERCISE 4.28. Show that the well defined isomorphism of vector spaces

$$\Lambda^{m_1} V_1 \otimes \Lambda^{m_2} V_2 \otimes \cdots \otimes \Lambda^{m_n} V_n \xrightarrow{\sim} W_{m_1,m_2,\dots,m_n}$$

is assigned by prescription $\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n \mapsto \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$, and verify that

$$\Lambda^k W = \bigoplus_{m_1,m_2,\dots,m_n} W_{m_1,m_2,\dots,m_n} \simeq \bigoplus_{m_1,m_2,\dots,m_n} \Lambda^{m_1} V_1 \otimes \Lambda^{m_2} V_2 \otimes \cdots \otimes \Lambda^{m_n} V_n .$$

We conclude that the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ can be identified with the component $W_{1,1,\dots,1} \subset \Lambda^n W$. Under this identification, the decomposable tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ go to the decomposable grassmannian monomials $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. Therefore, the Segre variety from n° 4.1.2 on p. 39 is the intersection of the grassmannian variety $\mathrm{Gr}(n, W) \subset \mathbb{P}(\Lambda^n W)$ with the projective subspace $\mathbb{P}\left(W_{1,1,\dots,1}\right) \subset \mathbb{P}(\Lambda^n W)$. In particular, the Segre variety is indeed an algebraic variety described by the system of quadratic equations from the Proposition 4.4 on p. 56 restricted onto the linear subspace $W_{1,1,\dots,1} \subset \Lambda^n W$.

Comments to some exercises

EXRC. 4.3. The first statement is verified by the same arguments as in **??** on p. **??** and n° 2.5.1. To prove the second, chose some dual bases $u_1, u_2, \dots, u_n \in U$, $u_1^*, u_2^*, \dots, u_n^* \in U^*$ and a basis $w_1, w_2, \dots, w_m \in W$. Then mn decomposable tensors $u_i^* \otimes w_j$ form a basis in $U^* \otimes V$. The matrix of operator

$$u_i^* \otimes w_j : u_k \mapsto \begin{cases} w_j & \text{for } k = i \\ 0 & \text{otherwise} \end{cases}.$$

has 1 in the crossing of j th row with i th column and zeros elsewhere. Thus, these operators span $\operatorname{Hom}(U, W)$.

EXRC. 4.4. For any linear mapping $f : V \to A$ the multiplication

$$V \times V \times \cdots \times V \to A,$$

which takes $(v_1, v_2, \dots, v_n)$ to their product $\varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n) \in A$, is multilinear. Hence, for each $n \in \mathbb{N}$ there exists a unique linear mapping $V^{\otimes n} \to A$ taking tensor multiplication to multiplication in A. Add them all together and get required algebra homomorphism $\mathsf{T}V \to A$ extending f. Since any algebra homomorphism $\mathsf{T}V \to A$ that extends f has to take $v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n)$, it coincides with the extension just constructed. Uniqueness of free algebra is proved exactly like the Lemma 4.1 on p. 39.

EXRC. 4.5. Since the decomposable tensors span $V^{*\otimes n}$ and the equality

$$i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v, w_1, w_2, \dots, w_{n-1})$$

is bilinear in v, φ, it is enough to check it for the decomposable $\varphi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$.

EXRC. 4.6. Fix a basis $e_1, \dots, e_p, u_1, \dots, u_q, w_1, \dots, w_r, v_1, \dots, v_s$ in V such that e_i form a basis in $U \cap W$, u_j and w_k extend it to some bases in U, W, and v_m complete everything to a basis in V. Then expand t through the standard monomial basis of $\mathsf{T}V$ built from this basis of V.

EXRC. 4.8. Fo all $v, w \in V$ we have

$$0 = \varphi(\dots, (v + w), \dots, (v + w), \dots) = \varphi(\dots, v, \dots, w, \dots) + \varphi(\dots, w, \dots, v, \dots).$$

Vice versa, if $\operatorname{char} \Bbbk \neq 2$, then $\varphi(\dots, v, \dots, v, \dots) = -\varphi(\dots, v, \dots, v, \dots)$ forces

$$\varphi(\dots, v, \dots, v, \dots) = 0.$$

EXRC. 4.9. See, e.g., the Proposition 11.2 on p. 260 in the sec. 11.2.2 of the book: A. L. Gorodentsev, *Algebra I. Textbook for Students of Mathematics.*, Springer, 2016.

EXRC. 4.10. Every multilinear map $\varphi : V \times V \times \cdots \times V \to W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $F : V^{\otimes n} \to W$ is linear. Such F is factorized through the projection $V^{\otimes n} \twoheadrightarrow S^n V$ if and only if

$$F(\cdots \otimes v \otimes w \otimes \cdots) = F(\cdots \otimes w \otimes v \otimes \cdots).$$

The latter is equivalent to $\varphi(\dots, v, w, \dots) = \varphi(\dots, w, v, \dots)$. This proves the universality of the multiplication in SV. Every linear map $f : V \to A$ induces the symmetric multilinear map $V \times V \times \cdots \times V \to A$, $(v_1, v_2, \dots, v_n) \mapsto \prod \varphi(v_i)$ for any $n \in \mathbb{N}$. The latter gives the linear map

$S^n V \to A$. All together these maps extend f to the homomorphism of $\Bbbk$-algebras $SV \to A$. Vice versa, every homomorphism of $\Bbbk$-algebras $SV \to A$, which extends f, takes $\prod v_i \to \prod \varphi(v_i)$ and coincides with the previous extension. The uniqueness of extension is verified as in the Lemma 4.1 on p. 39.

EXRC. 4.11. The first follows from $0 = (v+w)\otimes(v+w) = v\otimes w + w\otimes v$, the second from $v\otimes v + v\otimes v = 0$.

EXRC. 4.12. Similar to **??** on p. **??**.

EXRC. 4.13. If $\dim V = d$, then $Z(\Lambda V) = \Lambda^d V + \bigoplus_k \Lambda^{2k} V$. For even d, the first summand is contained in the second, for odd d the sum is direct.

EXRC. 4.15. Use that $\det A = \det A^t$, and transpose everything.

EXRC. 4.16. The summands form one S_n-orbit. The stabilizer of an element in this orbit consists of $m_1!\, m_2! \cdots m_d!$ independent permutations of coinciding factors. Hence, the length of orbit equals $\frac{n!}{m_1!m_2!\cdots m_d!}$.

EXRC. 4.17. For $v = \sum \alpha_i e_i$, the complete contraction of $v^{\otimes n}$ with $\widetilde{f} = \frac{m_1!\cdot m_2!\cdots m_d!}{n!} x_{[m_1,m_2,\dots,m_d]}$ is the sum of $n!/(m_1!\cdot m_2! \cdots m_d!)$ mutually equal products

$$\frac{m_1!\cdot m_2!\cdots m_d!}{n!}\cdot x_1(v)^{m_1}\cdot x_2(v)^{m_2}\cdot \cdots \cdot x_d(v)^{m_d} = \frac{m_1!\cdot m_2!\cdots m_d!}{n!}\cdot \alpha_1^{m_1}\alpha_2^{m_2}\cdots \alpha_d^{m_d}.$$

Thus, it coincides with the result of substitution $(x_1,x_2,\dots,x_n) = (\alpha_1,\alpha_2,\dots,\alpha_n)$ in the monomial $\frac{n!}{m_1!m_2!\cdots m_d!} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$.

EXRC. 4.18. Use the same arguments as in the proof of multinomial expansion formula

$$(v_1+v_2+\cdots+v_k)^n = \sum_{m_1 m_2 \dots m_k} \frac{n!}{m_1!\,m_2!\cdots m_k!}\cdot v_1^{m_1} v_2^{m_2}\cdots v_k^{m_k}.$$

EXRC. 4.20. Since the Leibniz rule is linear in v, f, g, it is enough to check it for $v = e_i$, $f = x_1^{m_1}\dots x_d^{m_d}$, $g = x_1^{k_1}\dots x_d^{k_d}$. In this case it follows directly from the definition of polar map. The formula for $\widetilde{f}(v_1,v_2,\dots,v_n)$ follows from the equality $\widetilde{f}(v_1,x,\dots,x) = \frac{1}{n}\cdot \partial_{v_1} f(x)$ by induction in $n = \deg f$.

EXRC. 4.23. Similar to the Exercise 4.20.

EXRC. 4.24. Let $e_1,e_2,\dots,e_m$ be a basis in U. If $\omega \notin \Lambda^m U$, then the expansion of ω as a linear combination of basis monomials e_I contains a monomial whose index I differs from the whole $1, 2, \dots, m$. Let $k \notin I$. Then $e_k \wedge \omega \neq 0$, because the basis monomial $e_{\{k\}\sqcup I}$ appears in $e_k\wedge\omega$ with a nonzero coefficient. Conversely, if $\omega \in \Lambda^m U$, then $\omega = \lambda\cdot e_1\wedge e_2\wedge\cdots\wedge e_m$ and $e_i \wedge \omega = 0$ for all i.

EXRC. 4.26. See the Example 4.9 on p. 57.

EXRC. 4.27. Let $U \neq W$ be two subspaces of dimension m. Chose a basis

$$e_1,e_2,\dots,e_r,\, u_1,u_2,\dots,u_{m-r},\, w_1,w_2,\dots,w_{m-r},\, v_1,v_2,\dots,v_{d+r-2m} \in V$$

such that $e_1,e_2,\dots,e_r$ is a basis of $U\cap W$, vectors $u_1,u_2,\dots,u_{m-r}$ and $w_1,w_2,\dots,w_{m-r}$ complete it to bases in U and W respectively, and the remaining vectors are complementary to $U+W$. The Plücker embedding (**??**) sends U and V to the different basis monomials

$$v_1\wedge\cdots\wedge v_r\wedge u_1\wedge\cdots\wedge u_{m-r} \neq v_1\wedge\cdots\wedge v_r\wedge w_1\wedge\cdots\wedge w_{m-r}$$

in $\Lambda^m V$.