§7 Affine algebraic geometry

We assume on default in §7 that the ground field $\Bbbk$ is algebraically closed.

7.1 Affine Algebraic – Geometric dictionary. A map $\varphi : X \to Y$ between affine algebraic varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ is called *regular* or *polynomial* if its action is described in coordinates by the rule $(x_1, x_2, \dots, x_n) \mapsto (\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x))$, where $\varphi_i(x) \in \Bbbk[x_1, x_2, \dots, x_n]$. We write $\mathcal{Aff}_{\Bbbk}$ for the category[1] of affine algebraic varieties and regular maps between them.

7.1.1 Coordinate algebra. A function $f : X \to \Bbbk$ on an affine algebraic variety $X \subset \mathbb{A}^n$ is called *regular* if it provides X with a regular map $f : X \to \mathbb{A}^1$, that is, if there exists some polynomial in the coordinates $x_1, x_2, \dots, x_n$ on $\mathbb{A}^n$ whose restriction on X coincides with f. Two polynomials determine the same regular function if and only if they are congruent modulo the ideal $I(X) = \{f \in \Bbbk[x_1, x_2, \dots, x_n] \mid f|_X \equiv 0\}$. The regular functions $X \to \Bbbk$ form a $\Bbbk$-algebra with respect to the usual addition and multiplication of functions taking values in a field. This algebra is called the *coordinate algebra* of X and denoted by

$$\Bbbk[X] \overset{\text{def}}{=} \operatorname{Hom}_{\mathcal{Aff}_{\Bbbk}}(X, \mathbb{A}^1) \simeq \Bbbk[x_1, x_2, \dots, x_n] / I(X). \tag{7-1}$$

Since for a function $f : X \to \Bbbk$, the equality $f^n = 0$ implies $f = 0$, the coordinate algebra $\Bbbk[X]$ has no nilpotent elements. This forces the ideal $I(X)$ to be *radical*, that is, coinciding with $\sqrt{I(X)}$. Algebras without nilpotent elements are said to be *reduced*. We write $\mathcal{Alg}_{\Bbbk}$ for the category of finitely generated reduced $\Bbbk$-algebras and $\Bbbk$-algebra homomorphisms respecting units.

PROPOSITION 7.1
Every reduced finitely generated algebra A over an algebraically closed field $\Bbbk$ is isomorphic to the coordinate algebra $\Bbbk[X]$ of some affine algebraic variety X over $\Bbbk$.

PROOF. Write A as a quotient $A = \Bbbk[x_1, x_2, \dots, x_n] / J$. Since A is reduced, $\sqrt{J} = J$. By the strong Nullstellensatz, this forces J to coincide with the ideal $I(V(J))$ of the affine algebraic variety $V(J) \subset \mathbb{A}^n$. Thus, $A = \Bbbk[X]$ for $X = V(J)$. □

7.1.2 Maximal spectrum. Associated with every point $p \in X$ on an affine algebraic variety X is the *evaluation homomorphism* $\operatorname{ev}_p : \Bbbk[X] \to \Bbbk$, $f \mapsto f(p)$. It is obviously surjective and therefore, its kernel

$$\mathfrak{m}_p \overset{\text{def}}{=} \ker \operatorname{ev}_p = \{f \in \Bbbk[X] \mid f(p) = 0\}$$

is a maximal ideal in $\Bbbk[X]$, called the *maximal ideal of the point* $p \in X$. Note that for every $g \in \Bbbk[X]$, the residue class $g \pmod{\mathfrak{m}_p}$ coincides in $\Bbbk[X]/\mathfrak{m}_p \simeq \Bbbk$ with the class of constant $g(p)$, i.e., the evaluation at p can be thought as the factorization modulo the ideal $\mathfrak{m}_p \subset \Bbbk[X]$.

Given an arbitrary commutative $\Bbbk$-algebra A, the set of all maximal ideals $\mathfrak{m} \subset A$ is called the *maximal spectrum* of A and denoted by $\operatorname{Spec}_{\mathrm{m}}(A)$. For every $\mathfrak{m} \in \operatorname{Spec}_{\mathrm{m}} A$, the quotient $A/\mathfrak{m} \supset \Bbbk$

[1] A *category* $\mathcal{C}$ is a class of *objects*, where for every ordered pair of objects X, Y, a set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* $X \to Y$ is given and for every ordered triple of objects X, Y, Z the *composition map*

$$\operatorname{Hom}_{\mathcal{C}}(Y, Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{C}}(X, Z), \quad (\varphi, \psi) \mapsto \varphi \circ \psi,$$

is defined such that $(\eta \circ \varphi) \circ \psi = \eta \circ (\varphi \circ \psi)$ for any composable morphisms η, φ, ψ, and every object X possesses the *identity endomorphism* $\operatorname{Id}_X \in \operatorname{Hom}_{\mathcal{C}}(X, X)$ satisfying the relations $\varphi \circ \operatorname{Id}_X = \varphi$ and $\operatorname{Id}_X \circ \psi = \psi$ for all morphisms $\varphi : X \to Y$, $\psi : Z \to X$.

is a field. If A is finitely generated, then this quotient is finitely generated as well and therefore, is an algebraic extension of $\Bbbk$ by Theorem 6.2 on p. 75. For algebraically closed $\Bbbk$, this forces $A/\mathfrak{m} = \Bbbk$. Thus, for such A and $\Bbbk$, every factorization homomorphism $A \twoheadrightarrow A/\mathfrak{m} = \Bbbk$ takes values in $\Bbbk$. Vice versa, every homomorphism of $\Bbbk$-algebras $\varphi : A \to \Bbbk$ sends 1 to 1 and therefore, is surjective. Thus, its kernel $\ker\varphi$ is a maximal ideal in A. We conclude that for an arbitrary finitely generated algebra over an algebraically closed field $\Bbbk$, the $\Bbbk$-algebra homomorphisms $A \to \Bbbk$ stay in canonical bijection with the points of $\operatorname{Spec}_{\mathrm{m}} A$. In what follows, we make no difference between the points $\mathfrak{m} \subset \operatorname{Spec}_{\mathrm{m}} A$ and the homomorphisms $A \to \Bbbk$, and write $\mathrm{ev}_{\mathfrak{m}} : A \to \Bbbk$ for the factorization homomorphism modulo $\mathfrak{m}$. There is a natural homomorphism from A to the algebra $A \to \Bbbk^{\operatorname{Spec}_{\mathrm{m}} A}$ of functions $\operatorname{Spec}_{\mathrm{m}} A \to \Bbbk$. It sends an element $a \in A$ to the function

$$a : \operatorname{Spec}_{\mathrm{m}} A \to \Bbbk\,, \quad \mathfrak{m} \mapsto \mathrm{ev}_{\mathfrak{m}}(a) = a \,(\mathrm{mod}\ \mathfrak{m}) \in A/\mathfrak{m} = \Bbbk\,. \tag{7-2}$$

The kernel of this homomorphism, that is, the set of all elements $a \in A$ vanishing at every point of the spectrum, coincides with the intersection of all maximal ideals in A. It is called the *Jackobson radical* of A and denoted $\mathfrak{r}(A)$.

Proposition 7.2
For a finitely generated algebra A over an algebraically closed field $\Bbbk$, the Jackobson radical $\mathfrak{r}(A)$ coincides with the set of all nilpotent elements in A, that is, with the *nilradical*

$$\mathfrak{n}(A) \overset{\text{def}}{=} \sqrt{0} = \{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}\,.$$

Exercise 7.1. Check that $\mathfrak{n}(A)$ is an ideal in A.

Proof of Proposition 7.2. Since the algebra of functions $\operatorname{Spec}_{\mathrm{m}} A \to \Bbbk$ is reduced, every nilpotent element of A produces the zero function. Thus, $\mathfrak{n}(A) \subset \mathfrak{r}(A)$. To prove the converse inclusion, let $A_{\mathrm{red}} \overset{\text{def}}{=} A/\mathfrak{n}(A)$. Since A_{red} is finitely generated and reduced, there exists an affine algebraic variety $X \subset \mathbb{A}^n$ with the coordinate algebra $\Bbbk[X] = A_{\mathrm{red}}$. If a lies in the kernel of every homomorphism $A \to \Bbbk$, then the image of a in $\Bbbk[X]$ also lies in the kernel of every homomorphism $\Bbbk[X] \to \Bbbk$. In particular, $a(p) = 0$ for all $p \in X$, that is, $a = 0$ in $\Bbbk[X] = A/\mathfrak{n}(A)$. Hence, $a \in \mathfrak{n}(A)$. □

Exercise 7.2. For an arbitrary commutative ring A with unit, show that the nilradical $\mathfrak{n}(A)$ coincides with the intersection of all prime[1] ideals in A.

Proposition 7.3
For an affine algebraic variety X over an algebraically closed field $\Bbbk$, the map

$$X \to \operatorname{Spec}_{\mathrm{m}} \Bbbk[X]\,, \quad p \mapsto \mathfrak{m}_p = \ker \mathrm{ev}_p\,,$$

is bijective.

Proof. This map is injective regardless of whether $\Bbbk$ is algebraically closed, because for $p \neq q$, there exists, for example, an affine linear function $f : \mathbb{A}^n \to \Bbbk$ such that $f(p) = 0$ and $f(q) = 1$. Let us show that over algebraically closed field $\Bbbk$, every maximal ideal $\mathfrak{m} \subset \Bbbk[X]$ coincides with

[1] An ideal $\mathfrak{p} \subset A$ is called *prime*, if the quotient ring $A/\mathfrak{p}$ has no zero divisors.

$\mathfrak{m}_p = \ker \mathrm{ev}_p$ for some $p \in X$. Write $\widetilde{\mathfrak{m}} \subset \Bbbk[x_1, x_2, \dots, x_n]$ for the full preimage of $\mathfrak{m}$ under the factorization homomorphism $\Bbbk[x_1, x_2, \dots, x_n] \twoheadrightarrow \Bbbk[X] = \Bbbk[x_1, x_2, \dots, x_n]/I(X)$. Since

$$\Bbbk[x_1, x_2, \dots, x_n]/\widetilde{\mathfrak{m}} = \Bbbk[X]/\mathfrak{m} = \Bbbk,$$

$\widetilde{\mathfrak{m}}$ is a proper maximal ideal containing $I(X)$. By the week Nullstellensatz, $V(\widetilde{\mathfrak{m}})$ is a non-empty subset of X. Pick a point $p \in V(\widetilde{\mathfrak{m}})$. Since $\mathfrak{m} \subset \mathfrak{m}_p$ and $\mathfrak{m}$ is maximal, $\mathfrak{m} = \mathfrak{m}_p$. □

EXAMPLE 7.1 (THE AFFINE SPACE)
Since a homomorphism of algebras $\varphi : \Bbbk[x_1, x_2, \dots, x_n] \to \Bbbk$ is uniquely determined by the images of generators $\varphi(x_i) \in \Bbbk$, a bijection $\mathrm{Spec_m}\, \Bbbk[x_1, x_2, \dots, x_n] \xrightarrow{\sim} \mathbb{A}^n$ is given by sending φ to the point $p = \big(\varphi(x_1), \dots, \varphi(x_n)\big) \in \mathbb{A}^n$. As a consequence, we conclude that every maximal ideal in $\Bbbk[x_1, x_2, \dots, x_n]$ is generated by an n-tuple of linear forms $x_i - p_i$, where $p_i \in \Bbbk$, $1 \leqslant i \leqslant n$, and the equality of ideals $\big(x_1 - p_1, \dots, x_n - p_n\big) = \big(x_1 - q_1, \dots, x_n - q_n\big)$ is equivalent to the equality of points $(p_1, p_2, \dots, p_n) = (q_1, q_2, \dots, q_n)$ in $\mathbb{A}^n$.

7.1.3 Pullback homomorphisms. Associated with an arbitrary map of sets $\varphi : X \to Y$ is the *pullback homomorphism* $\varphi^* : \Bbbk^Y \to \Bbbk^X$, which maps a function $f : Y \to \Bbbk$ to the composition

$$f \circ \varphi : X \to \Bbbk.$$

Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic varieties with the coordinate algebras

$$\Bbbk[X] = \Bbbk[x_1, x_2, \dots, x_n]/I(X), \quad \Bbbk[Y] = \Bbbk[y_1, y_2, \dots, y_m]/I(Y),$$

and let the map $\varphi : X \to Y$ be given in coordinates by the assignment

$$(x_1, x_2, \dots, x_n) \mapsto (\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)).$$

Then the pullbacks of the coordinate functions $y_i : Y \to \Bbbk$ are $\varphi^*(y_i) = \varphi_i$. Since the y_i generate the coordinate algebra $\Bbbk[Y]$, the regularity of φ, meaning that $\varphi_i(x) \in \Bbbk[x_1, x_2, \dots, x_n]$, is equivalent to the inclusion $\varphi^*(\Bbbk[Y]) \subset \Bbbk[X]$, meaning that the pullback of every regular function is regular.

EXERCISE 7.3. Verify that a set-theoretical map of topological spaces $X \to Y$ is continuous if and only if the pullback of every continuous function on Y is a continuous function on X.

Note that the inclusion of sets $\varphi(X) \subset Y$ implies the inclusion of ideals $\varphi^*\big(I(Y)\big) \subset I(X)$, which forces the map $\Bbbk[y_1, y_2, \dots, y_m] \to \Bbbk[x_1, x_2, \dots, x_n]$, $y_i \mapsto \varphi_i(x_1, x_2, \dots, x_n)$, to be correctly factorized through the map $\Bbbk[Y] = \Bbbk[y_1, y_2, \dots, y_m]/I(Y) \to \Bbbk[x_1, x_2, \dots, x_n]/I(X) = \Bbbk[X]$. Thus, every regular map of affine algebraic varieties $\varphi : X \to Y$ produces the well defined pullback homomorphism of the coordinate algebras $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$.

Vice versa, associated with every homomorphism of finitely generated $\Bbbk$-algebras $\psi : A \to B$ is the pullback map of spectra $\psi^* : \mathrm{Spec_m}\, B \to \mathrm{Spec_m}\, A$ which takes an evaluation $\mathrm{ev}_{\mathfrak{m}} : B \to \Bbbk$ with the kernel $\mathfrak{m} \in \mathrm{Spec_m}\, B$ to the evaluation $\mathrm{ev}_{\mathfrak{m}} \circ \psi = \mathrm{ev}_{\psi^{-1}(\mathfrak{m})} : A \to \Bbbk$ with the kernel $\psi^{-1}(\mathfrak{m}) \in \mathrm{Spec_m}\, A$.

PROPOSITION 7.4
For any affine algebraic varieties X, Y, the pullback maps

$$\mathrm{Hom}_{\mathcal{Aff}_{\Bbbk}}(X, Y) \underset{\psi^* \leftarrow\!\shortmid \psi}{\overset{\varphi \mapsto \varphi^*}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{Alg}_{\Bbbk}}(\Bbbk[Y], \Bbbk[X])$$

are inverse to each other and therefore bijective.

Proof. Let a regular map from $X \subset \mathbb{A}^n$ to $Y \subset \mathbb{A}^m$ act by the rule

$$(x_1, x_2, \dots, x_n) \mapsto (\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x))$$

for some $\varphi_i(x) \in \Bbbk[x_1, x_2, \dots, x_n]$. Then the pullback $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$ takes $y_i \mapsto \varphi_i \pmod{I(X)}$. The pullback of φ^*, that is, the map $\varphi^{**} : \operatorname{Spec_m} \Bbbk[X] \to \operatorname{Spec_m} \Bbbk[Y]$, sends the evaluation at a point $p = (p_1, p_2, \dots, p_n) \in X$

$$\mathrm{ev}_p : \Bbbk[X] \to \Bbbk, \quad f(x) \mapsto f(p),$$

to the composition $\mathrm{ev}_p \circ \varphi^*$, which sends every generator $y_i \in \Bbbk[Y]$ to $\varphi_i(p)$ and therefore, coincides with the evaluation at the point $\varphi(p)$. Thus, $\varphi^{**} = \varphi$. The equality $\psi^{**} = \psi$ for a homomorphism $\psi : \Bbbk[Y] \to \Bbbk[X]$ is checked similarly, and we leave its verification to the reader as an exercise. □

7.1.4 Equivalence of categories. A contravariant functor[1] $F : \mathcal{A}\!f\!f_\Bbbk \to \mathcal{A}\ell g_\Bbbk$ is assigned by sending an affine algebraic variety X to the coordinate algebra $\Bbbk[X]$ and a regular map of affine algebraic varieties $\varphi : X \to Y$ to the pullback homomorphism $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$.

By the Proposition 7.1, every algebra A in $\mathcal{A}\ell g_\Bbbk$ is isomorphic to the coordinate algebra of some affine algebraic variety. Let us fix such an isomorphism

$$f_A : A \xrightarrow{\sim} \Bbbk[X_A] \tag{7-3}$$

for each A, and for every affine algebraic variety X, put $X_{\Bbbk[X]} = X$ and $f_{\Bbbk[X]} : \Bbbk[X] \to \Bbbk[X]$ to be the identity map $\mathrm{Id}_{\Bbbk[X]}$. The pullback maps of the isomorphisms (7-3) assign the bijections $f_A^* : X_A \xrightarrow{\sim} \operatorname{Spec_m} A$. Write $P : \mathcal{A}\ell g_\Bbbk \to \mathcal{A}\!f\!f_\Bbbk$ for the contravariant functor sending an algebra A to the affine variety X_A and a homomorphism of algebras $\psi : A \to B$ to the regular map of algebraic varieties $P(\psi) = f_A^{*-1} \circ \psi^* \circ f_B^* : X_B \to X_A$, which fits in the commutative diagram

$$\begin{array}{ccc} X_B & \xrightarrow{P(\psi)} & X_A \\ {\scriptstyle f_B^*}\downarrow\wr & & \wr\downarrow{\scriptstyle f_A^*} \\ \operatorname{Spec}_m(B) & \xrightarrow[\psi^*]{} & \operatorname{Spec}_m(B), \end{array}$$

where the bottom row is the pullback of ψ.

Exercise 7.4. Convince yourself that $P(\psi)$ is a regular map of affine algebraic varieties.

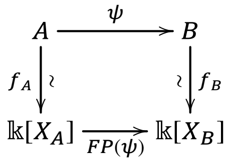
By the construction, the composition $P \circ F : \mathcal{A}\!f\!f_\Bbbk \to \mathcal{A}\!f\!f_\Bbbk$ acts identically on the objects and morphisms, that is, equals the identity functor. The reverse composition $F \circ P$ sends every algebra A to the isomorphic algebra $\Bbbk[X_A]$, and the isomorphisms (7-3) assign a *natural isomorphism*[2] between

[1]A *contravariant functor*[3] $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns an object $F(X)$ in $\mathcal{D}$ to every object X in $\mathcal{C}$, and assigns a map $\mathrm{Hom}_\mathcal{C}(X, Y) \to \mathrm{Hom}_\mathcal{D}(F(Y), F(X))$, $\varphi \mapsto F(\varphi)$, to every ordered pair of objects X, Y in $\mathcal{C}$, such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all objects X and $F(\varphi \circ \psi) = F(\psi) \circ F(\varphi)$ for all composable morphisms φ, ψ in $\mathcal{C}$.

[2]Two functors $F, G : \mathcal{C} \to \mathcal{D}$ are said to be *naturally isomorphic* if for every object X in $\mathcal{C}$ there exist an isomorphism $f_X : F(X) \xrightarrow{\sim} G(X)$ in $\mathcal{D}$ such that for every morphism $\varphi : X \to Y$ in $\mathcal{C}$, the following diagram in $\mathcal{D}$ is commutative:

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(\varphi)} & F(X) \\ {\scriptstyle f_Y}\downarrow\wr & & \wr\downarrow{\scriptstyle f_X} \\ G(Y) & \xrightarrow[G(\varphi)]{} & G(X) \end{array}$$

the identity functor on $\mathcal{A}\ell g_{\Bbbk}$ and the composition $F \circ P$. Indeed, for every homomorphism of algebras $\psi : A \to B$, the diagram



is commutative, because $FP(\psi) = F(f_A^{*-1} \circ \psi^* \circ f_B^*) = f_B^{**} \circ \psi^{**} \circ f_A^{**-1} = f_B \circ \psi \circ f_A^{-1}$.

In this situation, the functors F and P are said to be *contravariant equivalences* between the categories $\mathcal{A}\ell g_{\Bbbk}$ and $\mathcal{A}\!f\!f_{\Bbbk}$. Informally, this means that an affine algebraic variety X is recovered from the coordinate algebra $\Bbbk[X]$ uniquely up to a regular isomorphism, the regular morphisms $X \to Y$ stay in the canonical bijection with the homomorphisms of algebras $\Bbbk[Y] \to \Bbbk[X]$, this bijection respects the composition of morphisms and is respected by the isomorphisms of algebraic varieties sharing the same coordinate algebra.

A choice of isomorphisms (7-3) used in the construction of the functor P is equivalent to a presentation of every algebra A in the form $\Bbbk[x_1, x_2, \dots, x_n]/I(X_A)$, that is, to a choice of algebra generators for A. This is similar to a choice of basis in a vector space V, that provides V with an isomorphism $V \xrightarrow{\sim} \Bbbk^n$. Thus, the set $\operatorname{Spec}_m A$ can be thought of as an «abstract» affine algebraic variety which possesses various realizations in the form $V(I) \subset \mathbb{A}^n$ provided by a choice of presentation $A \xrightarrow{\sim} \Bbbk[x_1, x_2, \dots, x_n]/I$ of the algebra A in terms of generators and relations.

Example 7.2 (punctured line and hyperbola)

As we have seen in Example 7.1, the spectrum $\operatorname{Spec}_m \Bbbk[t]$ is realized as the affine line $\mathbb{A}^1 = \Bbbk$ by sending an evaluation $\psi : \Bbbk[t] \to \Bbbk$ to the point $p = \psi(t) \in \Bbbk$. By the same reason, the spectrum $\operatorname{Spec}_m \Bbbk[t, t^{-1}]$ of the algebra of Laurent polynomials is naturally identified with the punctured line $\mathbb{A}^1 \smallsetminus \{0\} = \Bbbk^*$, because the evaluations $\psi : \Bbbk[t, t^{-1}] \to \Bbbk$ also stay in bijection with their values $p = \psi(t) = 1/\psi(t^{-1}) \in \Bbbk^*$. A presentation of the algebra $\Bbbk[t, t^{-1}]$ in terms of generators and relations is provided by the isomorphism $f : \Bbbk[t, t^{-1}] \xrightarrow{\sim} \Bbbk[x, y]/(xy - 1)$, $t \mapsto x$, $t^{-1} \mapsto y$. It realizes $\operatorname{Spec}_m \Bbbk[t, t^{-1}]$ as the hyperbola $V(xy - 1) \subset \mathbb{A}^2$. The pullback map $f : V(xy - 1) \xrightarrow{\sim} \mathbb{A}^1 \smallsetminus \{0\}$ projects the hyperbola on the punctured x-axis along the y-axis in $\mathbb{A}^2$.

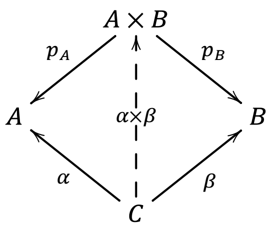


Fig. 7◇1. The universal property of product.

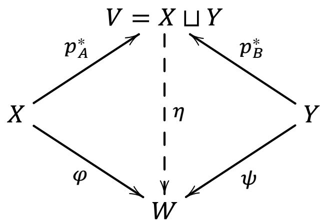


Fig. 7◇2. The universal property of coproduct.

Example 7.3 (coproduct of affine algebraic varieties)

The direct product of $\Bbbk$-algebras $A \times B$ is uniquely determined by the following universal property of the projections $p_A : A \times B \to A$ and $p_B : A \times B \to B$: for any pair of $\Bbbk$-algebra homomorphisms $\alpha : C \to A$ and $\beta : C \to B$ there exists a unique homomorphism of $\Bbbk$-algebras $\alpha \times \beta : C \to A \times B$ such that $p_A \circ (\alpha \times \beta) = \alpha$ and $p_B \circ (\alpha \times \beta) = \beta$, see fig. 7◇1.

Exercise 7.5. Convince yourself that if a pair of $\Bbbk$-algebra homomorphisms $p'_A : Z \to A$ and $p'_B : Z \to B$ also possesses this universal property, then the map $p'_A \times p'_B : Z \to A \times B$ is an isomorphism.

The direct product of finitely generated reduced $\Bbbk$-algebras $A = \Bbbk[X]$, $B = \Bbbk[Y]$ is also finitely generated and reduced. Hence, the spectrum $\operatorname{Spec}_{\mathrm{m}}(A \times B)$ is realized by an affine algebraic variety V equipped with the pullback maps $p_A^* : X \to V$, $p_B^* : Y \to V$ possessing the dual[1] universal property: for any pair of regular maps $\varphi : X \to W$, $\psi : X \to W$ of affine algebraic varieties there exists a unique regular map $\eta : V \to W$ such that $\eta \circ p_A^* = \varphi$, $\eta \circ p_B^* = \psi$, see fig. 7◇2. This universal property determines the variety V uniquely up to a unique regular isomorphism commuting with the maps p_A^*, p_B^*. In an abstract category, the object V possessing this universal property is called the *coproduct* of objects X, Y.

Exercise 7.6. Convince yourself that in the category of sets, the coproduct of sets X, Y is provided by the disjoint union $X \sqcup Y$, and verify that $\operatorname{Spec}_{\mathrm{m}}(A \times B) = \operatorname{Spec}_{\mathrm{m}} A \sqcup \operatorname{Spec}_{\mathrm{m}} B$ as a set.

Thus, the disjoint union $X \sqcup Y$ of affine algebraic varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, has a structure of affine algebraic variety whose coordinate algebra is isomorphic to $\Bbbk[X] \times \Bbbk[Y]$.

Example 7.4 (product of affine algebraic varieties)

The direct product of spectra $\operatorname{Spec}_{\mathrm{m}}(A) \times \operatorname{Spec}_{\mathrm{m}}(B)$ in the category of sets admits a structure of affine algebraic variety whose coordinate algebra is the *tensor product of algebras* $A \otimes B$, which gives the direct coproduct in the category $\mathcal{A}\ell g_{\Bbbk}$ and is constructed as follows. Let us equip the tensor product of vector spaces $A \otimes B$ over $\Bbbk$ with the multiplication defined by $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) \stackrel{\text{def}}{=} (a_1 a_2) \otimes (b_1 b_2)$.

Exercise 7.7. Verify that $A \otimes B$ becomes a commutative $\Bbbk$-algebra with the unit $1 \otimes 1$, and the $\Bbbk$-algebra homomorphisms $A \hookrightarrow A \otimes B \hookleftarrow B$, $a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$, give the coproduct in the category of commutative $\Bbbk$-algebras with unit.

It follows from the universal property of coproduct that there exists a bijection

$$\operatorname{Spec}_{\mathrm{m}}(A) \times \operatorname{Spec}_{\mathrm{m}}(B) \xrightarrow{\sim} \operatorname{Spec}_{\mathrm{m}}(A \otimes B)$$

sending a pair of homomorphisms $\mathrm{ev}_p : A \to \Bbbk$, $a \mapsto a(p)$ and $\mathrm{ev}_q : B \to \Bbbk$, $b \mapsto b(q)$, to the homomorphism $A \otimes B \to \Bbbk$, $a \otimes b \mapsto a(p)b(q)$. If the algebras A, B are finitely generated, say, by some elements $a_1, a_2, \dots, a_n \in A$, $b_1, b_2, \dots, b_m \in B$, then $A \otimes B$ is certainly generated by the elements $a_i \otimes b_j$. Let us verify that the tensor product of reduced algebras A, B is reduced. By Proposition 7.2 on p. 83, it is enough to check that every element $h \in A \otimes B$ that is evaluated to zero at every point of $\operatorname{Spec}_{\mathrm{m}}(A \otimes B)$ must be the zero element. Write such an element as $h = \sum f_\nu \otimes g_\nu$, where $g_\nu \in B$ are linearly independent over $\Bbbk$. Since $(\mathrm{ev}_p \otimes \mathrm{ev}_q)h = 0$ for all $(p, q) \in \operatorname{Spec}_{\mathrm{m}}(A \otimes B)$, the linear combination $\sum f_\nu(p) \cdot g_\nu \in B$ is the zero function on $\operatorname{Spec}_{\mathrm{m}} B$ for every fixed $p \in \operatorname{Spec}_{\mathrm{m}} A$. Since B is reduced, this linear combination is the zero element of B. Therefore, all its coefficients $f_\nu(p) = 0$, because of the linear independence of b_ν over $\Bbbk$. Since this holds for all $p \in \operatorname{Spec} A$, every element $f_\nu \in A$ is the zero function on $\operatorname{Spec}_{\mathrm{m}} A$. This forces $f_\nu = 0$, because A is reduced. Hence, $h = 0$. We conclude that the tensor product $\Bbbk[X] \otimes \Bbbk[Y]$ gives the direct coproduct in $\mathcal{A}\ell g_{\Bbbk}$. Thus, in the category of affine algebraic varieties, the direct product

$$\operatorname{Spec}_{\mathrm{m}}(A) \times \operatorname{Spec}_{\mathrm{m}}(B) = \operatorname{Spec}_{\mathrm{m}}\big(\Bbbk[X] \otimes \Bbbk[Y]\big)\,.$$

For example, $\mathbb{A}^n \times \mathbb{A}^m \simeq \mathbb{A}^{n+m}$, because of the isomorphism

$$\Bbbk[x_1, x_2, \dots, x_n] \otimes \Bbbk[y_1, y_2, \dots, y_m] \simeq \Bbbk[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m]$$

provided by the map $x_1^{s_1} x_2^{s_2} \dots x_n^{s_n} \otimes y_1^{r_1} y_2^{r_2} \dots y_m^{r_m} \mapsto x_1^{s_1} x_2^{s_2} \dots x_n^{s_n} y_1^{r_1} y_2^{r_2} \dots y_m^{r_m}$.

[1] That is, obtained from the original by reversing all arrows.

EXERCISE 7.8. Given some polynomial equations $f_\nu(x) = 0$, $g_\mu(y) = 0$, describing affine algebraic varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, write down an explicit system of polynomial equations whose solution set is $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m$.

7.2 Zariski topology. The set $X = \operatorname{Spec}_{\mathfrak{m}} A$ possesses the natural topology, called the *Zariski topology*, whose closed sets are the subsets of X that can be described by polynomial equations, i.e., the sets

$$\begin{aligned} V(I) &= \{x \in X \mid f(x) = 0 \text{ for all } f \in I\} = \\ &= \{\mathfrak{m} \in \operatorname{Spec}_{\mathfrak{m}} A \mid I \subset \mathfrak{m}\} = \\ &= \{\varphi : A \to \Bbbk \mid \varphi(I) = 0\} \end{aligned}$$

taken for all ideals $I \subset A$.

EXERCISE 7.9. Verify that A) $\varnothing = V((1))$ B) $X = V((0))$ C) $\bigcap_\nu V(I_\nu) = V\left(\sum_\nu I_\nu\right)$, where the ideal $\sum_\nu I_\nu$ consists of finite sums of elements $f_\nu \in I_\nu$ D) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$, where the ideal $IJ \subset I \cap J$ consist of finite sums of products ab with $a \in I$, $b \in J$.

The Zariski topology has a purely algebraic nature. It reflects divisibility relations rather than closeness or remoteness. For this reason some properties of the Zariski topology are discordant with intuition based on the metric topology. One of the most important differences which should be always taken in mind is that the Zarisky topology on the product $X \times Y$ is strictly finer than the product of Zariski topologies on the factors X, Y, i.e., the products of closed subsets in X, Y do not form a base for the closed subsets $Z \subset X \times Y$. For example, for $X = Y = \mathbb{A}^1$, every plane algebraic curve, e.g., the hyperbola $V(xy - 1)$, is Zariski closed in $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, whereas the products of closed subsets in $\mathbb{A}^1$ are exhausted by $\varnothing$, $\mathbb{A}^2$, and finite unions of points and lines parallel to the coordinate axes.

PROPOSITION 7.5 (BASE FOR OPEN SETS AND COMPACTNESS)
Every Zariski open subset U of an affine algebraic variety X is a finite union of *principal open sets*

$$\mathcal{D}(f) \stackrel{\text{def}}{=} X \smallsetminus V(f) = \{x \in X \mid f(x) \neq 0\}$$

for some $f \in \Bbbk[X]$, and is *compact* in the induced topology, meaning that every open covering of U contains a finite subcovering.

PROOF. Let $U = X \smallsetminus V(I)$. Since $\Bbbk[X]$ is Noetherian, $I = (f_1, f_2, \dots, f_m)$ for some $f_i \in \Bbbk[X]$. Therefore $V(I) = \bigcap V(f_i)$ and $U = \bigcup (X \smallsetminus V(f_i)) = \bigcup \mathcal{D}(f_i)$. Further, let U be covered by a family of principal open sets $\mathcal{D}(f_\nu)$, and I the ideal spanned by the functions f_ν. Then $V(I) \subset X \smallsetminus U$ and $I = (f_1, f_2, \dots, f_m)$ for some finite collection $f_1, f_2, \dots, f_m$ of the functions f_ν. Therefore, the open sets $\mathcal{D}(f_i)$, $1 \leqslant i \leqslant m$, cover U as well. □

PROPOSITION 7.6 (CONTINUITY OF REGULAR MAPS)
Every regular map of affine algebraic varieties $\varphi : X \to Y$ is continuous in the Zariski topology.

PROOF. For any closed set $V(I) \subset Y$, the preimage $\varphi^{-1}\big(V(I)\big)$ consists of the points $x \in X$ such that $0 = f(\varphi(x)) = \varphi^* f(x)$ for all $f \in I$. Therefore, it coincides with $V(J)$ for the ideal $J \subset \Bbbk[X]$ generated by the image of I under the pullback homomorphism $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$. □

7.2.1 Irreducible components. A topological space X is called *reducible* if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subsetneq X$. Otherwise X is called *irreducible*. In the usual metric topology, almost all spaces are reducible. In the Zariski topology, the irreducible affine algebraic varieties play the same role as the powers of prime numbers in arithmetic.

EXERCISE 7.10. Verify that $V(f) \subset X$ is nonempty and proper for any nonzero non-invertible element $f \in \Bbbk[X]$.

PROPOSITION 7.7
An affine algebraic variety X is irreducible if and only if its coordinate algebra $\Bbbk[X]$ has no zero divisors.

PROOF. If $X = X_1 \cup X_2$ with proper closed X_1, X_2, then there exist nonzero regular functions $f_1, f_2 \in \Bbbk[X]$ such that $f_1 \in I(X_1)$, $f_2 \in I(X_2)$. Since $f_1 f_2$ vanishes at every point of X, it equals zero in $\Bbbk[X]$. Conversely, if $f_1 f_2 = 0$ for some nonzero $f_1, f_2 \in \Bbbk[X]$, then $X = V(f_1) \cup V(f_2)$, where the closed sets $V(f_1)$, $V(f_2)$ are proper. □

COROLLARY 7.1
Given a polynomial $g \in \Bbbk[x_1, x_2, \dots, x_n]$, the affine hypersurface $V(g) \subset \mathbb{A}^n$ is irreducible if and only if $g = q^n$ for some irreducible $q \in \Bbbk[x_1, x_2, \dots, x_n]$ and $n \in \mathbb{N}$.

PROOF. Since the polynomial ring $\Bbbk[x_1, x_2, \dots, x_n]$ is a unique factorization domain, a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ is irreducible if and only if the quotient ring $\Bbbk[x_1, x_2, \dots, x_n] / (f)$ has no zero divisors, and for every f the radical $\sqrt{(f)}$ is the principal ideal generated by the product of all pairwise non-associated irreducible divisors of f. Therefore, $\Bbbk[V(f)] = \Bbbk[x_1, x_2, \dots, x_n] / \sqrt{(f)}$ has no zero divisors if and only if f has a unique (up to a constant factor) irreducible divisor. □

EXAMPLE 7.5 (BIG OPEN SETS)
If X is irreducible, then any two nonempty open sets $U_1, U_2 \subset X$ have nonempty intersection, because otherwise X could be decomposed as $X = (X \smallsetminus U_1) \cup (X \smallsetminus U_2)$. In other words, any nonempty open subset of an irreducible variety X is dense in X. Thus, the Zariski topology is quite far from being Hausdorf.

EXERCISE 7.11. Let X be an irreducible algebraic variety and $f, g \in \Bbbk[X]$. Prove that if $f(p) = g(p)$ for all points p from a nonempty open subset $U \subset X$, then $f = g$ in $\Bbbk[X]$.

THEOREM 7.1
Any affine algebraic variety X admits a decomposition $X = X_1 \cup X_2 \cup \dots \cup X_k$, where all $X_i \subset X$ are closed irreducible and $X_i \not\subset X_j$ for all $i \neq j$. This decomposition is unique up to renumbering of components.

PROOF. If X is reducible, write it as $X = Z_1 \cup Z_2$, where $Z_1, Z_2 \subset X$ are proper closed, and repeat the procedure recursively for every component until it stops on some finite decomposition $X = \bigcup Z_\nu$, where all Z_ν are irreducible. If the procedure newer stoped, we could chose an infinite strictly decreasing chain of closed sets $X \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots$, whose ideals form a strictly increasing chain $(0) \subsetneq I(Y_1) \subsetneq I(Y_2) \subsetneq \cdots$ in $\Bbbk[X]$, which is impossible, because $\Bbbk[X]$ is Noetherian. Now let $X_1 \cup X_2 \cup \dots \cup X_k = Y_1 \cup Y_2 \cup \dots \cup Y_m$ be two decompositions satisfying the conditions of the theorem. Since $Y_1 = \bigcup_i (Y_1 \cap X_i)$ is irreducible, $Y_1 \cap X_i = Y_1$ for some i, that is, $Y_1 \subset X_i$. By the same reason, $X_i \subset Y_j$ for some j. Since $Y_1 \not\subset Y_j$ for $j \neq 1$, we conclude that $Y_1 = X_i$. Renumber X_is in order to have $Y_1 = X_1$.

EXERCISE 7.12. Let $Z \subsetneq Y \subset X$ be closed, and Y irreducible. Prove that $Y = \overline{Y \smallsetminus Z}$ (the closure within X). Convince yourself that this may fail for reducible Y.

Now we can remove X_1 and Y_1, and proceed by induction on the number of components. □

DEFINITION 7.1
The decomposition $X = X_1 \cup X_2 \cup \ldots \cup X_k$ from Theorem 7.1 is called the *irreducible decomposition* of the algebraic variety X, and its components $X_i \subset X$ are called the *irreducible components* of X.

REMARK 7.1. (NOETHERIAN SPACES) Theorem 7.1 and its proof hold for any topological space X that does not allow strictly decreasing infinite chains of closed subsets $X \supsetneq Z_1 \supsetneq Z_2 \supsetneq \cdots$. Every such topological space is called *Noetherian*.

PROPOSITION 7.8
A nonzero element $f \in \Bbbk[X]$ is a zero divisor if and only if f has the zero restriction on some irreducible component of X.

PROOF. Let $fg = 0$ for some $g \neq 0$. Write $f_i, g_i \in \Bbbk[X_i]$ for the restrictions of f, g to the irreducible component $X_i \subset X$. Since $\Bbbk[X_i]$ has no zero divisors, at least one of f_i, g_i vanishes for every i. Since $g_i \neq 0$ for some i (otherwise $g = 0$ in $\Bbbk[X]$), we conclude that $f_i = 0$. Conversely, if $f_i = 0$, then $fg = 0$ for every nonzero function $g \in I\left(\bigcup_{\nu \neq i} X_\nu\right)$. □

7.3 Rational functions.

For every commutative ring A, the set of all non-zero-divisors

$$A^\circ \stackrel{\text{def}}{=} \{a \in A \mid ab \neq 0 \text{ for all } b \in A \smallsetminus 0\}$$

is *multiplicative*, i.e., contains 1, does not contain 0, and for $a, b \in A^\circ$, the product $ab \in A^\circ$. Thus, one can *localize* A with respect to A°, that is, consider the *fractions*[1] a/b with $a \in A$, $b \in A^\circ$. The fractions are added and multiplied by the standard rules and form a ring denoted by Q_A and called the *ring of fractions* of the commutative ring A. If A has no zero divisors, i.e., is a domain, then $A^\circ = A \smallsetminus 0$ and Q_A is a field, called *the field of fractions* of the domain A.

For an affine algebraic variety X, the $\Bbbk$-algebra of fractions $Q_{\Bbbk[X]}$ is traditionally denoted by $\Bbbk(X)$ and called the *algebra of rational functions* on X. Thus, a *rational function* on X is a fraction f/g, where $f, g \in \Bbbk[X]$ and g is not a zero divisor, and $f_1/g_1 = f_2/g_2$ in $\Bbbk(X)$ if and only if $f_1 g_2 = f_2 g_1$ in $\Bbbk[X]$. If X is irreducible, the algebra $\Bbbk(X)$ becomes a field.

A rational function $f \in \Bbbk(X)$ is said to be *regular* at a point $x \in X$ if there exist a fraction $g/h = f$ such that $h(x) \neq 0$. In this case, the number $f(x) \stackrel{\text{def}}{=} g(x)/h(x) \in \Bbbk$ is referred to as the *value* of f at the point $x \in X$.

EXERCISE 7.13. Verify that the value $f(x)$ does not depend on the choice of admissible representation $f = g/h$.

[1] Given a multiplicative set $S \subset A$, the fraction a/s with the denominator in S is the class of pair $(a, s) \in A \times S$ modulo the equivalence relation on $A \times S$ generated by the identifications $\frac{a}{s} = \frac{at}{st}$ for all $a \in A$, $s, t \in S$. It is a good exercise, to show that $a_1/s_1 = a_2/s_2$ if and only if $(a_1 s_2 - a_2 s_1)t = 0$ for some $t \in S$. The fraction can be added and multiplied by the usual rules, and form a commutative ring denoted by $S^{-1}A$ and called the *localization* of A with respect to S. See details in: A. L. Gorodentsev. Algebra I. Textbook for Students of Mathematics. Springer, 2016. Section 4.1.

If a rational function $f = g/h$ has $h(x) \neq 0$ at some point $x \in X$, then f is regular at every point in the principal open neighborhood $\mathcal{D}(h)$ of the point x. Moreover, by Proposition 7.8, this neighborhood has nonempty intersection with every irreducible component of X, because h is not a zero divisor in $\Bbbk[X]$. Therefore, all points $x \in X$, at which f is regular, form an open dense subset in X. It is called the *domain* of f and denoted $\mathrm{Dom}(f)$.

EXERCISE 7.14. Verify that $f_1 = f_2$ in $\Bbbk(X)$ if and only if $f_1(x) = f_2(x)$ for all x in some open dense subset of X.

PROPOSITION 7.9
Let X be an affine algebraic variety over an infinite field, and $f \in \Bbbk(X)$ a rational function. Then $I_f \overset{\text{def}}{=} \{g \in \Bbbk[X] \mid gf \in \Bbbk[X]\}$ is an ideal in $\Bbbk[X]$ with the zero set $V(I_f) = X \smallsetminus \mathrm{Dom}(f)$.

PROOF. The closed set $X \smallsetminus \mathrm{Dom}(f)$ is the set of common zeros of denominators $q \in \Bbbk[X]^\circ$ appearing in various fractional representations $f = p/q$. The intersection $I_f \cap \Bbbk[X]^\circ$ consists exactly of these denominators. It is enough to check that the intersection $I_f \cap \Bbbk[X]^\circ$ generates the ideal I_f. Let us show that it spans I_f even as a vector space over $\Bbbk$. By Proposition 7.8, the complement $I_f \smallsetminus \Bbbk[X]^\circ$, which consists of all zero divisors in I_f, splits in the finite union of vector subspaces $I_f \cap I(X_i)$. Since $I_f \cap \Bbbk[X]^\circ \neq \varnothing$, every subspace $I_f \cap I(X_i)$ is proper. If the $\Bbbk$-linear span of $I_f \cap \Bbbk[X]^\circ$ is proper too, the vector space I_f becomes a finite union of proper subspaces. The next exercise makes this impossible. □

EXERCISE 7.15. Prove that a vector space over an infinite field cannot be decomposed into a finite union of proper vector subspaces.

7.3.1 The structure sheaf. Given an affine algebraic variety X, for every open $U \subset X$, we put

$$\mathcal{O}_X(U) \overset{\text{def}}{=} \{f \in \Bbbk(X) \mid \mathrm{Dom}(f) \supset U\}.$$

The assignment $\mathcal{O}_X : U \mapsto \mathcal{O}_X(U)$ provides the topological space X with a sheaf[1] of $\Bbbk$-algebras, called the *structure sheaf* of X or the *sheaf of regular rational functions* on X. For an open $U \subset X$, the algebra $\mathcal{O}_X(U)$ is often denoted by $\Bbbk[U]$ and referred to as the *algebra of rational functions regular* in U. This makes no confusion for $U = X$, because of the following claim.

PROPOSITION 7.10
Let X be an affine algebraic variety over an algebraically closed field and $h \in \Bbbk[X]^\circ$. Then

$$\mathcal{O}_X(\mathcal{D}(h)) = \Bbbk[X][h^{-1}] = \{f/h^n \mid f \in \Bbbk[X],\ n \in \mathbb{Z}_{\geqslant 0}\}$$

is the localization of $\Bbbk[X]$ with respect to the multiplicative system of nonnegative integer powers h^n.

[1] A *presheaf* F of objects from a category $\mathcal{C}$ on a topological space X is a contravariant functor from the category of open subsets in X and inclusions of open sets as the morphisms to the category $\mathcal{C}$. This means that attached to every open $U \subset X$ is an object $F(U)$ in $\mathcal{C}$, called *sections* of F over U. Depending on $\mathcal{C}$, the sections can form a set, a vector space, an algebra, etc Associated with every inclusion $U \subset W$ of open sets is the morphism $F(W) \to F(U)$, called the *restriction of sections* from W to U. The restriction of a section $s \in F(W)$ to $U \subset W$ is commonly denoted $s|_U$. The functoriality of F means that for every triple of nestled open sets $U \subset V \subset W$ and every $s \in F(W)$, the relation $s|_U = s|_V\big|_U$ holds. A presheaf F is called a *sheaf*, if for every set of sections $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, there exists a unique section $s \in F\left(\bigcup_i U_i\right)$ such that $s|_{U_i} = s_i$ for all i.

PROOF. A rational function $f \in \Bbbk(X)$ is regular in $\mathcal{D}(h)$ if and only if $V(h)$ contains the closed subset $X \smallsetminus \mathrm{Dom}(f) = V(I_f)$, see Proposition 7.9. By the strong Nullstellensatz[1], $h^d \in I_f$ for some $d \in \mathbb{N}$. Thus, $h^d \cdot f \in \Bbbk[X]$, as required. □

COROLLARY 7.2
$\mathcal{O}_X(X) = \Bbbk[X]$.

PROOF. Apply Proposition 7.10 for $h = 1$, $\mathcal{D}(h) = X$. □

EXAMPLE 7.6 (PRINCIPAL OPEN SETS AS AFFINE ALGEBRAIC VARIETIES)
For every $h \in \Bbbk[X]^\circ$, the algebra $\mathcal{O}_X(\mathcal{D}(h)) = \Bbbk[X][h^{-1}] \simeq \Bbbk[X][t]/(1-ht)$ is finitely generated and reduced, and the points of the principal open set $\mathcal{D}(h) \subset X$ stay in bijection with the points of the hypersurface $V(1-ht) \subset X \times \mathbb{A}^1$. The notation $\Bbbk\big[\mathcal{D}(h)\big]$, which may be treated either as the coordinate algebra of the affine algebraic variety $\mathcal{D}(h) \subset X \times \mathbb{A}^1$ or as the subring in $\Bbbk(X)$ formed by the rational functions regular in the open set $\mathcal{D}(h) \subset X$, makes actually no confusion: two interpretations agree by Proposition 7.10. The pullback homomorphism of the projection

$$\pi : V(1-ht) \to X ,$$

which maps $V(1-ht) \subset X \times \mathbb{A}^1$ isomorphically to $\mathcal{D}(h) \subset X$, is the canonical map

$$\pi^* : \Bbbk[X] \hookrightarrow \Bbbk[X][h^{-1}], \quad f \mapsto f/1 ,$$

from a ring to its localization. By the universal property of the ring of fractions, this map is uniquely extended to the isomorphism

$$\widetilde{\pi}^* : \Bbbk(X) \xrightarrow{\sim} \Bbbk\big(\mathcal{D}(h)\big) . \tag{7-4}$$

CAUTION 7.1. A nonprincipal open set $U \subset X$ might not be an affine algebraic variety, and the canonical inclusion $U \hookrightarrow \mathrm{Spec}_{\mathrm{m}}\,\mathcal{O}_X(U)$, sending a point $u \in U$ to its maximal ideal $\mathfrak{m}_u = \ker \mathrm{ev}_u \subset$ $\subset \mathcal{O}_X(U)$, may be nonsurjective.

EXERCISE 7.16. Let $n \geqslant 2$ and $U = \mathbb{A}^n \smallsetminus O$ be the complement to the origin. Verify that $\mathcal{O}_{\mathbb{A}^n}[U] = \Bbbk[\mathbb{A}^n]$ and therefore, $\mathrm{Spec}_{\mathrm{m}}\,\mathcal{O}_{\mathbb{A}^n}[U] = \mathbb{A}^n \neq U$.

PROPOSITION 7.11
Let $X = X_1 \cup X_2 \cup \ldots \cup X_k$ be the irreducible decomposition of an affine algebraic variety X. Then $\Bbbk(X) = \Bbbk(X_1) \times \Bbbk(X_2) \times \ldots \times \Bbbk(X_k)$.

PROOF. Write $I = I\big(\bigcup_{i \neq j}(X_i \cap X_j)\big) \subset \Bbbk[X]$ for the ideal of all regular functions on X vanishing on every intersection $X_i \cap X_j$, $i \neq j$.

EXERCISE 7.17. Prove that I is linearly spanned over $\Bbbk$ by $I \cap \Bbbk[X]^\circ$.

Let us chose some regular function $f \in I \cap \Bbbk[X]^\circ$ and write $f_i = f \pmod{I(X_i)} \in \Bbbk[X_i]$ for its restriction to the irreducible component $X_i \subset X$. Then the affine algebraic variety

$$W = \mathcal{D}(f) = \mathrm{Spec}_{\mathrm{m}}\,\Bbbk[X][f^{-1}]$$

[1] See Theorem 6.3 on p. 78.

splits into a disjoint union of affine algebraic varieties

$$W_i = W \cap X_i = \mathcal{D}(f_i) = \operatorname{Spec}_{\mathrm{m}} \Bbbk[X_i][f_i^{-1}].$$

By Example 7.3, $\Bbbk[W] \simeq \Bbbk[W_1] \times \Bbbk[W_2] \times \cdots \times \Bbbk[W_k]$.

EXERCISE 7.18. For family of commutative rings A_ν, prove that $\left(\prod A_\nu\right)^\circ = \prod A_\nu^\circ$ as sets, and deduce from this the isomorphism $Q_{\prod A_\nu} \simeq \prod Q_{A_\nu}$ for the rings of fractions.

Therefore, $\Bbbk(X) \simeq \Bbbk(W) \simeq \prod \Bbbk(W_i) \simeq \prod \Bbbk(X_i)$ by formula (7-4). □

7.4 Geometric properties of algebra homomorphisms. Every homomorphism of finitely generated reduced $\Bbbk$-algebras $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$ can be canonically factorized into a composition of a quotient epimorphism followed by a monomorphism:

$$\Bbbk[Y] \xrightarrow{\varphi_1^*} \Bbbk[Y]/\ker(\varphi^*) = \operatorname{im}(\varphi^*) \xhookrightarrow{\varphi_2^*} \Bbbk[X]. \tag{7-5}$$

Since $\Bbbk[Y]$ is finitely generated and $\Bbbk[X]$ is reduced, the $\Bbbk$-algebra $\Bbbk[Y]/\ker(\varphi^*) = \operatorname{im}(\varphi^*) \subset \Bbbk[X]$ is both finitely generated and reduced. Thus, it is the coordinate algebra of the affine algebraic variety $Z = \operatorname{Spec}_{\mathrm{m}}(\operatorname{im}(\varphi^*)) \simeq V(\ker(\varphi^*)) \subset Y$. The injectivity of homomorphism $\varphi_1^* : \Bbbk[Z] \to \Bbbk[X]$ means that there are no nonzero functions $f \in \Bbbk[Z]$ vanishing on $\varphi_1(X) \subset Z$. Therefore, $\varphi_1(X)$ is *Zariski dense* in Z. In other words, $Z = \overline{\varphi(X)} \subset Y$ is the closure of $\varphi(X)$ in Y, situated within Y as the zero set $V(\ker \varphi^*)$ of the ideal $\ker \varphi^* \subset \Bbbk[Y]$. Thus, the algebraic factorization (7-5) geometrically corresponds to the factorization of a regular map of algebraic varieties $\varphi : X \to Y$ into the composition

$$X \xrightarrow{\varphi_2} Z = \overline{\varphi(X)} \xhookrightarrow{\varphi_1} Y$$

of the closed immersion $Z \hookrightarrow Y$ preceded by the regular morphism $X \to Z$ with dense image.

7.4.1 Closed immersions. A regular morphism of affine algebraic varieties $\varphi : X \to Y$ is called a *closed immersion* if its pullback homomorphism $\varphi^* : \Bbbk[Y] \to k[X]$ is surjective. In this case, φ establishes the regular isomorphism between X and the closed subset $V(\ker \varphi^*) \subset Y$. The pullback of this isomorphism of algebraic varieties is the canonical isomorphism of $\Bbbk$-algebras

$$\Bbbk[Y]/\ker \varphi^* \simeq \Bbbk[X].$$

For an irreducible closed subset $Z \subset X$, the pullback homomorphism $i^* : \Bbbk[X] \twoheadrightarrow \Bbbk[Z]$ of the closed immersion $i : Z \hookrightarrow X$ takes values in the integral domain $\Bbbk[Z]$, canonically embedded into its field of fractions $\Bbbk(Z)$. By the universal property of $\Bbbk(X)$, the epimorphism i^* is uniquely extended to the epimorphism

$$\operatorname{ev}_Z : \Bbbk(X) \twoheadrightarrow \Bbbk(Z), \tag{7-6}$$

which restricts the rational functions from X onto Z. Intuitively, the homomorphism (7-6) can be thought of as the evaluation of rational functions at the «generic point» of Z. The result of such evaluation is an element of $\Bbbk(Z)$, which may be further evaluated at particular points of Z. It follows from the surjectivity of homomorphism (7-6) that every rational function on Z is a restriction of some rational function on X, i.e. can be written as a fraction p/q whose denominator $q \in \Bbbk[X]^\circ$ is not a zero divisor in $\Bbbk[X]$. Note that such a presentation may be not so obvious in the case when $Z \subset X$ is an irreducible component of X.

EXERCISE 7.19. Let $X = V(xy) = \operatorname{Spec}_{\mathrm{m}} \Bbbk[x,y]/(xy)$ be the Cartesian cross on the affine plane $\mathbb{A}^2 = \operatorname{Spec}_{\mathrm{m}} \Bbbk[x,y]$, and $Z = \operatorname{Spec}_{\mathrm{m}} \Bbbk[x] = V(y)$ be its horizontal component. Write the rational function $1/x \in \Bbbk(Z)$ as a fraction $p/q \in \Bbbk(X)$, where $q \in \Bbbk[X]^\circ$.

7.4.2 Dominant morphisms. For an irreducible variety X, a regular morphism of algebraic varieties $\varphi : X \to Y$ is called *dominant* if its pullback homomorphism $\varphi^* : \Bbbk[Y] \to k[X]$ is injective. As we have seen above, this means that $\overline{\varphi(X)} = Y$. For reducible X, a regular map $\varphi : X \to Y$ is called *dominant* if its restriction $\varphi_i = \varphi|_{X_i}$ onto every irreducible component $X_i \subset X$ assigns the dominant map $\varphi_i : X_i \to Y$. In this case the pullback $\varphi_i^* : \Bbbk[Y] \hookrightarrow \Bbbk[X_i] \subset \Bbbk(X_i)$ embeds $\Bbbk[Y]$ in the *field* $\Bbbk(X_i)$. In particular, this forces Y to be irreducible. By the universal property of $\Bbbk(Y)$, the previous inclusion is uniquely extended to the inclusion of fields $\Bbbk(Y) \hookrightarrow \Bbbk(X_i)$. Thus, every dominant morphism $X = \bigcup X_i \to Y$ leads to the inclusion $\Bbbk(Y) \hookrightarrow \prod \Bbbk(X_i) = \Bbbk(X)$.

Exercise 7.20. Prove that any dominant morphism of irreducible affine algebraic varieties $\varphi : X \to Y$ can be factorized as

$$X \xhookrightarrow{\psi} Y \times \mathbb{A}^m \xrightarrow{\pi} Y , \tag{7-7}$$

where ψ is a closed immersion, and π is the projection along $\mathbb{A}^m$.

7.4.3 Finite morphisms. Every regular map of affine algebraic varieties $\varphi : X \to Y$ equips $\Bbbk[X]$ with the structure of a finitely generated algebra over the subring $\varphi^*(\Bbbk[Y]) = \Bbbk[\overline{\varphi(X)}] \subset \Bbbk[X]$. The map φ is called *finite* if $\Bbbk[X]$ is finitely generated as a *module*[1] over $\varphi^*(k[Y])$, or equivalently, if the extension of rings $\varphi^*(\Bbbk[Y]) \subset \Bbbk[X]$ is an integral extension.

Proposition 7.12 (closeness of finite morphisms)
Let $\varphi : X \to Y$ be a finite morphism of affine algebraic varieties, and $Z \subset X$ a closed subset. Then $\varphi(Z) \subset Y$ is also closed, and the restriction $\varphi|_Z : Z \to \varphi(Z)$ is a finite morphism. For irreducible X and proper $Z \subsetneq X$, the image $\varphi(Z) \subsetneq Y$ is also proper.

Proof. Write $I = I(Z) \subset \Bbbk[X]$ for the ideal of Z. The pullback homomorphism of the restricted map $\varphi|_Z : Z \to Y$ is factorized as $\varphi|_Z^* : \Bbbk[Y] \xrightarrow{\varphi^*} \Bbbk[X] \twoheadrightarrow \Bbbk[X]/I$, where the second arrow is the quotient homomorphism. Since $\Bbbk[X]$ is finitely generated as $\varphi^*(\Bbbk[Y])$-module, the quotient $\Bbbk[Z] = \Bbbk[X]/I$ is finitely generated as a module over $\varphi|_Z^*(\Bbbk[Y]) = \varphi^*(\Bbbk[Y]) \,/\, \big(I \cap \varphi^*(\Bbbk[Y])\big)$. Therefore, the restricted map $\varphi|_Z : Z \to \overline{\varphi(Z)}$ is finite. The equality $\varphi(Z) = \overline{\varphi(Z)}$ can be proved separately for each irreducible component of Z. Thus, we can assume that $X = Z$ is irreducible, and $Y = \overline{Z}$. In this case, φ^* embeds $A = \Bbbk[Y]$ in $B = \Bbbk[X]$ as a subalgebra $A \subset B$, this extension of algebras is integral, B has no zero divisors, and the map φ from $X = \operatorname{Spec_m} B$ to $Y = \operatorname{Spec_m} A$ sends a maximal ideal $\mathfrak{m} \subset B$ to the intersection $\mathfrak{m} \cap A \in \operatorname{Spec_m} A$. We have to show that for every maximal ideal $\mathfrak{m} \subset A$, there exists a maximal ideal $\widetilde{\mathfrak{m}} \subset B$ such that $\widetilde{\mathfrak{m}} \cap A = \mathfrak{m}$. If the ideal $\mathfrak{m}B$, spanned by $\mathfrak{m}$ in B, is proper, then every maximal ideal $\widetilde{\mathfrak{m}} \subset B$ containing $\mathfrak{m}B$ solves the problem. It remains to check that $\mathfrak{m}B \neq B$ for every proper ideal $\mathfrak{m} \subset A$. Assume the contrary. Let $\mathfrak{m}B = B$ for some maximal ideal $\mathfrak{m} \subset A$, and $b_1, b_2, \dots, b_m \in B$ span B as a A-module. Then $(b_1, b_2, \dots, b_m) = (b_1, b_2, \dots, b_m) \cdot M$ for some $m \times m$ matrix with elements in $\mathfrak{m}$. Hence, $(b_1, b_2, \dots, b_m) \cdot (E - M) = 0$. Similarly to the prove of Lemma 6.2 on p. 72, this implies that the multiplication by $\det(E - M)$ annihilates B, because it acts on the generators as

$$(b_1, b_2, \dots, b_m) \mapsto (b_1, b_2, \dots, b_m) \cdot (\det(E - M) \cdot E) = (b_1, b_2, \dots, b_m) \cdot (E - M)(E - M)^\vee ,$$

[1] That is, there are some $f_1, f_2, \dots, f_m \in \Bbbk[X]$ such that any $h \in \Bbbk[X]$ can be written as $h = \sum \varphi^*(g_i) f_i$ for appropriate $g_i \in \Bbbk[Y]$.

where $(E - M)^\vee$ is the adjunct matrix for $(E - M)$. Since B has no zero divisors, $\det(E - M) = 0$. Expanding the determinant shows that $1 \in \mathfrak{m}$, i.e., the ideal $\mathfrak{m} \subset A$ is not proper. Contradiction.

To prove the last statement of the Proposition, consider a nonzero function $f \in \Bbbk[X]$ whose restriction to $Z \subsetneq X$ is zero. It satisfies some polynomial equation with coefficients in $\varphi^*(k[Y])$. Let

$$\varphi^*(g_0)f^m + \varphi^*(g_1)f^{m-1} + \cdots + \varphi^*(g_{m-1})f + \varphi^*(g_m) = 0$$

be such an equation of the minimal possible degree. Then $g_m \neq 0$, because otherwise the degree could be decremented by canceling[1] one f. Evaluation of the left hand side at all points $z \in Z$ shows that $\varphi^*(g_m)|_Z = g_m|_{\varphi(Z)} = 0$. Hence, $\varphi(Z) \subset V(g_m) \subsetneq Y$ is proper. □

7.4.4 Normal varieties. An irreducible affine algebraic variety Y is called *normal* if its coordinate algebra $\Bbbk[Y]$ is a normal ring in the sense of n° 6.3. This means that $\Bbbk[Y]$ is integrally closed in the field of rational functions $\Bbbk(Y)$. Since every factorial ring is normal, every irreducible affine variety with the factorial coordinate algebra is normal. For example, the affine space $\mathbb{A}^n$ is normal for every n.

Proposition 7.13 (openness of finite surjection onto normal variety)
Let Y be a normal affine algebraic variety. Then every finite regular surjection $\varphi : X \twoheadrightarrow Y$ is open[2] Moreover, for all closed irreducible subsets $Z \subset Y$, every irreducible component of $\varphi^{-1}(Z)$ is surjectively mapped onto Z.

Proof. Since $\varphi^* : \Bbbk[Y] \hookrightarrow \Bbbk[X]$ is injective, we can consider $\Bbbk[Y]$ as a subalgebra in $\Bbbk[X]$. It is enough to show that φ maps any principal open set $\mathcal{D}(f) \subset X$ to an open subset of Y. This means that for every point $p \in \mathcal{D}(f)$, there exists a regular function $a \in \Bbbk[Y]$ such that $\varphi(p) \in \mathcal{D}(a) \subset \varphi(\mathcal{D}(f))$ in Y. To construct such a function, consider the map

$$\psi = \varphi \times f : X \to Y \times \mathbb{A}^1, \quad p \mapsto \big(\varphi(p), f(p)\big).$$

Its pullback homomorphism $\psi^* : \Bbbk[Y \times \mathbb{A}^1] = \Bbbk[Y][t] \to k[X]$ evaluates polynomials in t with coefficients in $\Bbbk[Y]$ at the element $f \in \Bbbk[X]$. Write μ_f for the minimal polynomial of f over $\Bbbk(Y)$. By Corollary 6.4, the coefficients of μ_f belong to $\Bbbk[Y]$. This forces ψ^* to be the factorization homomorphism modulo the principal ideal $(\mu_f) = \ker \psi^* \subset \Bbbk[Y \times \mathbb{A}^1]$. Thus, ψ is the finite surjection of X onto the hypersurface in $Y \times \mathbb{A}^1$ defined by the equation $\mu_f = 0$. Let us write $\mu_f = \mu_f(y;t)$ as the polynomial in the coordinate t on $\mathbb{A}^1$ with the coefficients $a_i \in \Bbbk[Y]$:

$$\mu_f = t^m + a_1(y)t^{m-1} + \cdots + a_m(y) \in \Bbbk[Y][t] = \Bbbk[Y \times \mathbb{A}^1].$$

The restriction of μ_f onto the line $y \times \mathbb{A}^1$ over a point $y \in Y$ is the polynomial in t whose roots are equal to the values of f at all points of X mapped to y by φ. In particular, $\varphi(\mathcal{D}(f))$ consists of those $y \in Y$ over which the polynomial $\mu(y;t)$ has a non-zero root. Since the polynomial $\mu_f(\varphi(p);t)$ that appears for $y = \varphi(p)$ has the root $f(p) \neq 0$, at least one of the coefficients of μ_f, say $a_k(y)$, does not vanish at $y = \varphi(p)$. This forces the polynomial $\mu_f(q;t)$ to have a nonzero root for all $q \in \mathcal{D}(a_k)$. Hence, $\mathcal{D}(a_k) \subset \varphi(\mathcal{D}(f))$ as required.

To prove the second statement, consider the irreducible decomposition $\pi^{-1}(Z) = C_1 \cup \ldots \cup C_m$ and let $U_i = X \smallsetminus \bigcup_{\nu \neq i} C_\nu$, $W_i = U_i \cap C_i = C_i \smallsetminus \bigcup_{\nu \neq i} C_\nu$. Since U_i is open in X, its image $\varphi(U_i)$ is open in

[1]This can be done, because $\Bbbk[X]$ has no zero divisors.
[2]That is, $\varphi(U)$ is open in Y for any open $U \subset X$.

Y, and therefore $Z \cap \varphi(U_i) = \varphi(W_i)$ is open and dense within Z, because Z is irreducible. By the same reason, W_i is dense in C_i. Therefore, $\varphi(C_i) = \varphi(\overline{W}_i) = \overline{\varphi(W_i)} = \overline{Z \cap \varphi(U_i)} = Z$. □

Comments to some exercises

Exrc. 7.1. If $a^n = 0$ and $b^m = 0$, then $(a+b)^{m+n-1} = 0$ and $(ca)^n = 0$ for all c.

Exrc. 7.2. Since $A/\mathfrak{p}$ has no zero divisors for all prime $\mathfrak{p} \subset A$, every factorization map $A \twoheadrightarrow A\mathfrak{p}$ by prime $\mathfrak{p}$ annihilates all the nilpotents. Thus, $\mathfrak{n}(A) \subset \bigcap \mathfrak{p}$. Conversely, let $a \in A$ be non-nilpotent. Then all nonnegative integer powers a^m form the multiplicative system A. Write $A[a^{-1}]$ for the localization[1] by this system. This is a nonzero ring[2]. The full preimage of any prime ideal[3] $\mathfrak{m} \subset A[a^{-1}]$ under the canonical homomorphism $A \to A[a^{-1}]$ is the prime ideal of A that does not contain a.

Exrc. 7.6. Homomorphisms $\Bbbk[X] \times \Bbbk[Y] \twoheadrightarrow \Bbbk$ stay in bijection with the pairs of homomorphisms $\Bbbk[X] \twoheadrightarrow \Bbbk$, $\Bbbk[Y] \twoheadrightarrow \Bbbk$.

Exrc. 7.7. Since $(a_1a_2)\otimes(b_1b_2)$ is linear in each of four elements, the multiplication $(a_1\otimes b_1)\cdot(a_2\otimes b_2) \overset{\text{def}}{=} (a_1a_2)\otimes(b_1b_2)$ is correctly extended to the $\Bbbk$-bilinear map $(A\otimes B)\times(A\otimes B) \to A\otimes B$, which provides $A \otimes B$ with a commutative associative binary operation. The required universal property of maps $A \xrightarrow{\alpha} A \otimes B \xleftarrow{\beta} B$ follows from the universal property of the tensor product of vector spaces. Namely, for any two homomorphisms of $\Bbbk$-algebras with unit $\varphi : A \to C$, $\psi : B \to C$, the bilinear map $A\times B \to C$, $(a,b) \mapsto \varphi(a)\cdot\psi(b)$, is uniquely passed through the tensor product $A\otimes B$.

Exrc. 7.8. Take the union of equations $f_\nu(x) = 0$, $g_\mu(y) = 0$, each considered as the equation on the whole set of coordinates (x, y) in $\mathbb{A}^n \times \mathbb{A}^m$.

Exrc. 7.9. The equalities (a), (b), (c), and the inclusions $V(I)\cup V(J) \subset V(I\cap J) \subset V(IJ) \subset V(I)\cup V(J)$ in (d) follow immediately from the definitions. Note that coincidence $V(I\cap J) = V(IJ)$ is equivalent to the equality of radicals $\sqrt{I\cap J} = \sqrt{IJ}$, which can be easily verified independently.

Exrc. 7.10. Let $X \subset \mathbb{A}^n$, $f \in \Bbbk[x_1, x_2, \dots, x_n]$. If $V(f) = X$, then $f \in I(X)$ and therefore, the class of f in $\Bbbk[X]$ equals zero. If $V(f) = \varnothing$, then the ideal spanned in $\Bbbk xn$ by f and $I(X)$ has empty zero set and therefore, contains the unity. Hence, $1 \equiv fg \pmod{I(X)}$ for some $g \in \Bbbk[x_1, x_2, \dots, x_n]$. Thus, the classes of f and g are inverse one to the other in $\Bbbk[X]$.

Exrc. 7.11. Otherwise $X = (X \smallsetminus U) \cup V(f-g)$. More scientifically, this holds because both f, g are continuous and U is dense.

Exrc. 7.12. $Y = (Y \cap Z) \cup \overline{Y \smallsetminus Z}$, where the first subset of Y is proper by the assumption.

Exrc. 7.15. Let $V = U_1 \cup U_2 \cup \ldots \cup U_m$. For every i, chose a nonzero linear form $\xi_i \in V^*$ annihilating U_i. Then $f = \prod_{i=1}^m \xi_i \in S^mV^*$ is the nonzero polynomial on V evaluated to zero at every point of $\mathbb{A}(V)$. This is impossible over an infinite ground field.

Exrc. 7.16. Use the open covering $U = \bigcup \mathcal{D}(x_i)$ and Proposition 7.10.

Exrc. 7.17. Every intersection $I \cap I(X_i)$ is a proper vector subspace of I, because if $I \subset I(X_\nu)$, then $X_\nu \subset \bigcup_{i\neq j}(X_i\cap X_j)$ and therefore, $X_\nu \subset X_i\cap X_j$ for some $i \neq j$, although such inclusions are forbidden. If the $\Bbbk$-linear span of $I \cap \Bbbk[X]^\circ$ is proper too, I splits in a finite union of proper vector subspaces.

Exrc. 7.20. Let $A = \Bbbk[X]$, $B = \Bbbk[Y]$. The inclusion $\varphi^* : B \hookrightarrow A$ provides A with the structure of finitely generated B-algebra. This allows to rewrite A as $A \simeq B[x_1, x_2, \dots, x_m]/J$. Then

[1] See Section 4.1.1 of Algebra II.
[2] which may be a field
[3] which is zero if $A[a^{-1}]$ is a field

$\psi^* : B[x_1, x_2, \dots, x_m] \twoheadrightarrow A$ is the quotient homomorphism, and $\pi^* : B \hookrightarrow B[x_1, x_2, \dots, x_m]$ is the inclusion of constants into polynomial ring.