Algebraic Geometry

A Start Up Course

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This is a geometric introduction to the algebraic geometry. I hope to acquaint the readers with some basic figures underlying the modern algebraic technique and show how to translate things from the infinitely rich (but quite intuitive) world of figures to the scanty and finite (but very explicit) language of formulas. These lecture notes contain a lot of exercises crucial for understanding the subject. Some of them are commented at the end of book.

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§1 Projective geometry

1.1 Preliminaries. Algebraic geometry deals with figures looking locally[1] as a set of solutions for some system of polynomial equations on affine space. Recall briefly what does the latter mean.

1.1.1 Polynomials. Let V be a vector space of dimension n over a field $\Bbbk$. Its *dual space* V^* is the space of all linear maps $V \to \Bbbk$, also known as *linear forms* or *covectors*. We write $\langle\, \varphi\,,\, v\,\rangle = \varphi(v) \in \Bbbk$ for the value of a covector $\varphi \in V^*$ on a vector $v \in V$. Given a basis $e_1, e_2, \dots, e_n \in V$, its *dual basis* $x_1, x_2, \dots, x_n \in V^*$ consists of the coordinate linear forms defined by prescriptions

$$\langle\, x_i\,,\, e_j\,\rangle = \begin{cases} 1 & \text{if } \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

We write $SV^* = \Bbbk[x_1, x_2, \dots, x_n]$ for the algebra of polynomials in x_i's with coefficients in $\Bbbk$. Another choice of basis in V^* leads to an isomorphic algebra whose generators are obtained from x_i's by invertible linear change of variables. We write $S^dV^* \subset SV^*$ for the subspace of homogeneous polynomials of degree d. This subspace is not changed under linear changes of variables. A basis of S^dV^* is formed by the monomials $x_1^{m_1}x_2^{m_2}\dots x_n^{m_n}$ numbered by the collections $m = (m_1, m_2, \dots, m_n)$ of integers $0 \leqslant m_i \leqslant d$ such that $\sum m_i = d$.

Exercise 1.1. Make sure that $\dim S^dV^* = \binom{n+d-1}{d}$ as soon $\dim V = n$.

Remark 1.1. Actually, the *symmetric algebra* SV^* and *symmetric powers* S^dV^* of a vector space V^* admit an intrinsic coordinate-free definition, see n° 4.3.1 on p. 44 below. The algebra SV^* is *graded*, i.e.,

$$SV^* = \bigoplus_{d\geqslant 0} S^dV^*$$

as a vector space and $S^kV^* \cdot S^mV^* \subset S^{k+m}V^*$.

1.1.2 Affine space and polynomial functions. Associated with a vector space V of dimension n is the *affine space* $\mathbb{A}^n = \mathbb{A}(V)$, also called the *affinization* of V. By the definition, the points of $\mathbb{A}(V)$ are the vectors of V. The point corresponding to the zero vector is called the *origin* and denoted O. All the other points can be imagined as the heads of non zero radius-vectors drawn from the origin. Every polynomial $f = \sum_m a_m x_1^{m_1}\dots x_n^{m_n} \in SV^*$ produces the *polynomial function*

$$f : \mathbb{A}(V) \to \Bbbk\,, \quad v \mapsto \sum_m a_m \langle\, x_1\,,\, v\,\rangle^{m_1} \dots \langle\, x_n\,,\, v\,\rangle^{m_n}, \tag{1-1}$$

which evaluates the polynomial at the coordinates of points $v \in \mathbb{A}(V)$. Despite Proposition 1.1 below, this function is traditionally denoted by the same letter as polynomial.

Proposition 1.1
The homomorphism of algebras $\varepsilon : \Bbbk[x_1, x_2, \dots, x_n] \to \{\text{functions } \mathbb{A}^n \to \Bbbk\}$, which sends a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ to the corresponding polynomial function $f : \mathbb{A}^n \to \Bbbk$, is injective if and only if the ground field $\Bbbk$ is infinite.

[1] That is, in some neighbor of every point.

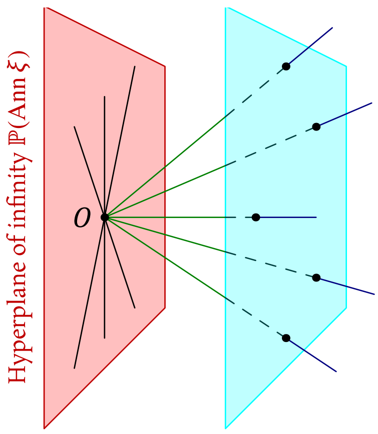
PROOF. If $\Bbbk$ consists of q elements, then the space of all functions $\mathbb{A}^n \to \Bbbk$ consists of q^{q^n} elements whereas the polynomial algebra $\Bbbk[x_1, x_2, \dots, x_n]$ is an infinite set. Hence, homomorphism ε is not injective. Let $\Bbbk$ be infinite. For $n = 1$, any non zero polynomial $f \in \Bbbk[x_1]$ has at most $\deg f$ roots. Hence, the corresponding polynomial function $f : \mathbb{A}^1 \to \Bbbk$ is not the zero function. For $n > 1$, we proceed inductively. Expand $f \in \Bbbk[x_1, x_2, \dots, x_n]$ as[1] $f(x_1, \dots, x_n) = \sum_k f_\nu(x_1, \dots, x_{n-1}) \cdot x_n^k$. If the polynomial function $f : \mathbb{A}^n \to \Bbbk$ vanishes identically, then the evaluation of all coefficients f_k at any point $w \in \mathbb{A}^{n-1} \subset \mathbb{A}^n$ turns f into polynomial $f(w, x_n) \in \Bbbk[x_n]$ that produces the zero function on line $\mathbb{A}^1 \subset \mathbb{A}^n$ passing through w parallel to x_n-axis. Hence, $f(w, x_n) = 0$ in $\Bbbk[x_n]$, i.e., all the coefficients $f_k(w)$ are identically zero functions of $w \in \mathbb{A}^{n-1}$. By induction, they all are the zero polynomials. □

EXERCISE 1.2. Let p be a prime number, $\mathbb{F}_p = \mathbb{Z}/(p)$ the residue field modulo p. Give an explicit example of non-zero polynomial $f \in \mathbb{F}_p[x]$ that produces the zero function $f : \mathbb{F}_p \to \mathbb{F}_p$.

1.1.3 Affine algebraic varieties. For a polynomial $f \in SV^*$, the set of all zeros of the corresponding polynomial function $f : \mathbb{A}(V) \to \Bbbk$ is denoted $V(f) \stackrel{\text{def}}{=} \{p \in \mathbb{A}(V) \mid f(p) = 0\}$ and called *an affine algebraic hypersurface*. An intersection of affine hypersurfaces is called *an affine algebraic variety*. Thus, an algebraic variety is a figure $X \subset \mathbb{A}^n$ defined by an arbitrary system of polynomial equations. The simplest example of a hypersurface is an *affine hyperplane* given by linear equation $\varphi(v) = c$, where $\varphi \in V^*$ is a non-zero linear form, and $c \in \Bbbk$. Such a hyperplane passes through the origin if and only if $c = 0$. In this case the hyperplane coincides with the affinization $\mathbb{A}(\operatorname{Ann}\varphi)$ of the vector subspace $\operatorname{Ann}(\varphi) = \{v \in V \mid \varphi(v) = 0\}$, annihilated by the covector φ. In general case, an affine hyperplane $\varphi(v) = c$ is the shift of $\mathbb{A}(\operatorname{Ann}\varphi)$ by an arbitrary vector u such that $\varphi(u) = c$.

1.2 Projective space. Much more interesting geometric object associated with a vector space V is the *projective space* $\mathbb{P}(V)$, also called the *projectivization* of V. By the definition, the points of $\mathbb{P}(V)$ are the vector subspaces of dimension one in V or, equivalently, the lines in $\mathbb{A}(V)$ passing through the origin. To see them as «usual dots» we have to intersect these lines with a screen, an affine hyperplane non-passing through the origin, like on fig. 1◇1. We write U_ξ for such the hyperplane given by linear equation $\xi(v) = 1$, where $\xi \in V^* \smallsetminus 0$, and call it the *affine chart* provided by covector ξ.

EXERCISE 1.3. Convince yourself that the map $\xi \mapsto U_\xi$ establishes a bijection between the non zero covectors and affine hyperplanes in $\mathbb{A}(V)$ that do not pass through the origin.



Affine chart $U_\xi = \{v \mid \xi(v) = 1\}$

Fig. 1◇1. Projective word.

No affine chart covers the whole $\mathbb{P}(V)$. The difference $\mathbb{P}(V) \smallsetminus U_\xi = \mathbb{P}(\operatorname{Ann}\xi)$ consists of all lines annihilated by ξ, i.e., laying inside the parallel copy of U_ξ drawn through the origin. The projective space formed by these lines is called the *infinity* of affine chart U_ξ.

Every point of $\mathbb{P}(V)$ is covered by some affine chart. For $\dim V = n + 1$, the charts are affine spaces of dimension n, and $\mathbb{P}(V)$ is looking locally as $\mathbb{A}^n$. By this reason, we say that $\mathbb{P}(V)$ has

[1] That is, as a polynomial in x_n with coefficients in the ring $\Bbbk[x_1, x_2, \dots, x_{n-1}]$

dimension n if $\dim V = n+1$, and write $\mathbb{P}_n$ instead of $\mathbb{P}(V)$ when the nature of V is not essential. Note that in a contrast with $\mathbb{A}^n = \mathbb{A}^1 \times \cdots \times \mathbb{A}^1$, the space $\mathbb{P}_n$ is not a direct product of n copies of $\mathbb{P}_1$. It follows from fig. 1⋄1 that $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{P}_{n-1}$ (a disjoint union). If we repeat this for $\mathbb{P}_{n-1}$ and further, we get the decomposition $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \ldots \sqcup \mathbb{A}^0$, where $\mathbb{A}^0 = \mathbb{P}_0$ is the one point set.

EXERCISE 1.4. Consider this decomposition over the finite field $\mathbb{F}_q$ of q elements and compute the cardinalities of both sides independently. Do you recognize the obtained identity on q?

1.2.1 Homogeneous coordinates. A choice of basis $\xi_0, \xi_1, \ldots, \xi_n \in V^*$ identifies V with $\Bbbk^{n+1}$ by sending $v \in V$ to $(\xi_0(v), \xi_1(v), \ldots, \xi_n(v)) \in \Bbbk^{n+1}$. Two coordinate rows $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ represent the same point $p \in \mathbb{P}(V)$ if and only if they are proportional, i.e., $x_\mu : x_\nu = y_\mu : y_\nu$ for all $0 \leqslant \mu \neq \nu \leqslant n$, where the identities of type $0 : x = 0 : y$ and $x : 0 = y : 0$ are allowed as well. Thus, the points $p \in \mathbb{P}(V)$ stay in bijection with the collections of ratios $(x_0 : x_1 : \ldots : x_n)$. The latter are called *homogeneous coordinates* on $\mathbb{P}(V)$ with respect to the chosen basis.

1.2.2 Local affine coordinates. Pick an affine chart $U_\xi = \{v \in V \mid \xi(v) = 1\}$ on $\mathbb{P}_n = \mathbb{P}(V)$. Any n covectors $\xi_1, \xi_2, \ldots, \xi_n \in V^*$ such that $\xi, \xi_1, \xi_2, \ldots, \xi_n$ form a basis of V^* provide U_ξ with *local affine coordinates*. Namely, consider the basis $e_0, e_1, \ldots, e_m$ in V dual to $\xi, \xi_1, \xi_2, \ldots, \xi_n$, and the affine coordinate system with origin at $e_0 \in U_\xi$ and axes $e_1, e_2, \ldots, e_n \in \operatorname{Ann} \xi$. The affine coordinates of a point $p \in \mathbb{P}_n$ in this system are computed as follows: rescale p to get the vector $u_p = p/\xi(p) \in U_\xi$ and evaluate n linear forms ξ_ν, $1 \leqslant \nu \leqslant n$, at this vector. The resulting numbers $(t_1(p), t_2(p), \ldots, t_n(p))$, where $t_i(p) = \xi_i(u_p) = \xi_i(p)/\xi(p)$ are called *local affine coordinates* of p in the chart U_ξ with respect to the covectors ξ_i. Note that local affine coordinates are *non-linear* functions of homogeneous coordinates.

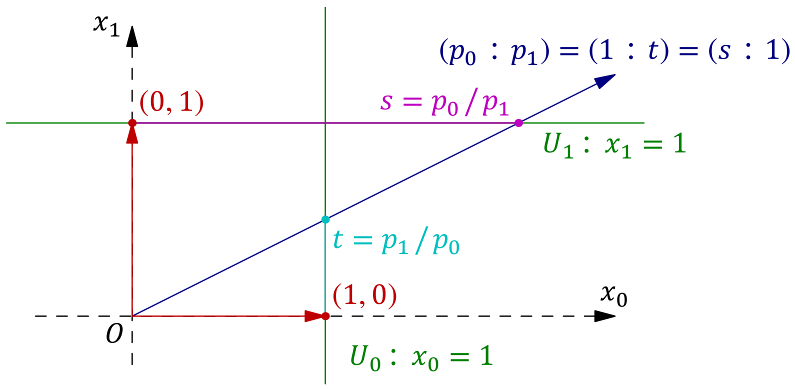


Fig. 1⋄2. The standard affine charts on $\mathbb{P}_1$.

EXAMPLE 1.1 (PROJECTIVE LINE)
The projective line $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ is covered by two affine charts $U_0 = U_{x_0}$ and $U_1 = U_{x_1}$ represented by the affine lines $x_0 = 1$ and $x_1 = 1$ in $\mathbb{A}^2 = \mathbb{A}(\Bbbk^2)$, see fig. 1⋄2. The chart U_0 covers the whole $\mathbb{P}_1$ except for the point $(0 : 1)$, the vertical axis in $\Bbbk^2$. The function $t = x_1|_{U_0} = x_1/x_0$ can be taken as a local affine coordinate in U_0. The infinite point of the chart U_1 is $(1 : 0)$, the horizontal axis in $\Bbbk^2$. The function $s = x_0|_{U_1} = x_0/x_1$ can be taken as a local affine coordinate in U_1. If a point $p = (p_0 : p_1) = (1 : p_1/p_0) = (p_0/p_1 : 1)$ is visible in the both charts, then its coordinates $t = p_1/p_0$ and $s = p_0/p_1$ are inverse to one other. Thus, $\mathbb{P}_1$ is obtained by gluing two distinct

copies of $\mathbb{A}^1 = \Bbbk$ along the complements to zero by the rule: a point s of the first $\mathbb{A}^1$ is identified with the point $1/s$ of the second. Over the field $\mathbb{R}$ of real numbers, this gluing procedure can be visualized as follows. Consider the circle of diameter one and identify two copies of $\mathbb{R}$ with two tangent lines passing through a pair of opposite points of the circle, see fig. 1⋄3. Then map each line to the circle via the central projection from the point opposite to the point of contact. It is immediate from fig. 1⋄3 that $1 : s = t : 1$ for any two points s, t of different lines mapped to the same point of the circle.

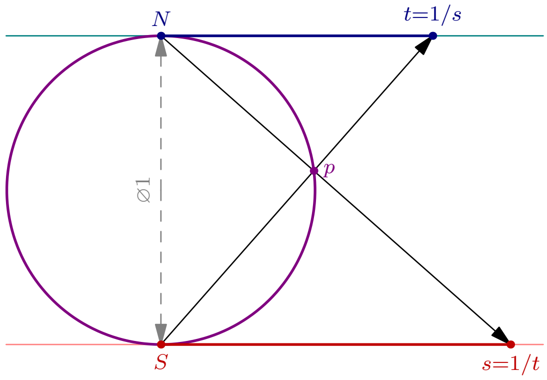


Fig. 1⋄3. $\mathbb{P}_1(\mathbb{R}) \simeq S^1$.

The same construction works for the field $\mathbb{C}$ of complex numbers as well, see fig. 1⋄4. Consider the sphere of diameter one and identify two copies of $\mathbb{C}$ with two tangent planes drown through the south and north poles of the sphere in the way[1] shown on fig. 1⋄4. The central projection of each plane to the sphere from the pole opposite to the point of contact sends complex numbers s, t, laying on different planes, to the same point of sphere if and only if s and t have opposite arguments and inverse absolute values[2], i.e., $t = 1/s$. Thus, the complex projective line can be thought of as the sphere.

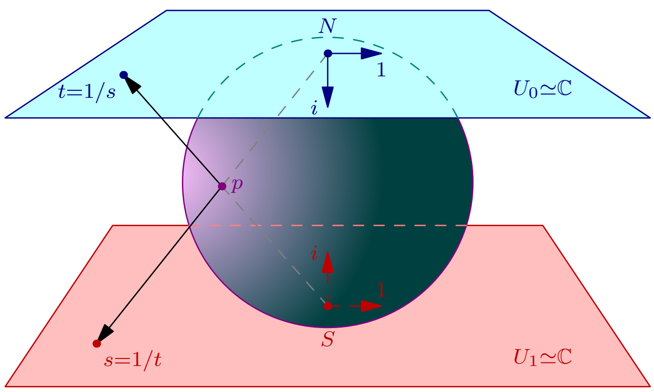


Fig. 1⋄4. $\mathbb{P}_1(\mathbb{C}) \simeq S^2$.

[1]Note that the both planes have compatible orientatons with respect to the sphere in the sense that they can be obtained one from the other by continuous move along the surface of sphere.

[2]The latter follows from fig. 1⋄3.

EXERCISE 1.5. Make sure that A) the real projective plane $\mathbb{P}_2(\mathbb{R})$ can be obtained by gluing a Möbius tape with a disc along their boundary circles[1] B) the real projective 3D space $\mathbb{P}_3 = \mathbb{P}(\mathbb{R}^4)$ can be identified with the Lie group $SO_3(\mathbb{R})$ of rotations of the Euclidean space $\mathbb{R}^3$ about the origin.

EXAMPLE 1.2 (STANDARD AFFINE COVERING FOR $\mathbb{P}_n$)
The standard affine covering of $\mathbb{P}_n = \mathbb{P}\left(\Bbbk^{n+1}\right)$ is formed by $n+1$ affine charts $U_\nu \overset{\text{def}}{=} U_{x_\nu} \subset \Bbbk^{n+1}$ given by equations $x_\nu = 1$. For every $\nu = 0, 1, \dots, n$, the functions

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}, \quad 0 \leqslant i \leqslant n,\ i \neq \nu,$$

are taken as default local affine coordinates inside U_ν. This allows to think of $\mathbb{P}_n$ as the result of gluing $n+1$ distinct copies $U_0, U_1, \dots, U_n$ of affine space $\mathbb{A}^n$ along their actual intersections inside $\mathbb{P}_n$. In terms of homogeneous coordinates $x = (x_0 : x_1 : \dots : x_n)$ on $\mathbb{P}_n$, the intersection $U_\mu \cap U_\nu$ consists of all $x \in \Bbbk^{n+1}$ such that $x_\mu \neq 0$ and $x_\nu \neq 0$. In terms of local affine coordinates inside U_μ and U_ν respectively, this locus is described by inequalities $t_\nu^{(\mu)} \neq 0$ and $t_\mu^{(\nu)} \neq 0$. Two points $t^{(\mu)} \in U_\mu$ and $t^{(\nu)} \in U_\nu$ are glued together in $\mathbb{P}_n$ if and only if $t_\nu^{(\mu)} = 1/t_\mu^{(\nu)}$ and $t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)}$ for $i \neq \mu, \nu$. The right hand sides of these relations are called the *transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$.

1.3 Projective algebraic varieties. Let us fix some basis $x_0, x_1, \dots, x_n$ in V^*. In a contrast with the affine geometry, a non-constant polynomial $f \in \Bbbk[x_0, x_1, \dots, x_n]$ does not produce a well defined function on $\mathbb{P}(V)$ anymore, since typically $f(v) \neq f(\lambda v)$ for non zero $v \in V$ and $\lambda \in \Bbbk$. However, for any *homogeneous* polynomial $f \in S^d V^*$, the zero set $V(f) = \{ p \in \mathbb{P}(V) \mid f(v) = 0 \}$ is still well defined in $\mathbb{P}(V)$, because $f(v) = 0 \iff f(\lambda v) = \lambda^d f(v) = 0$. In other words, for such f, the affine hypersurface $V(f) \subset \mathbb{A}(V)$ is a cone ruled by lines passing through the origin. The set of these lines is also denoted by $V(f) \subset \mathbb{P}(V)$ and called *a projective hypersurface* of degree $d = \deg f$. An intersection of projective hypersurfaces is called *an algebraic projective variety*.

The simplest example of a projective variety is *a projective subspace* $\mathbb{P}(U) \subset \mathbb{P}(V)$, the projectivization of a vector subspace $U \subset V$. It is described by a system of linear homogeneous equations $\varphi(v) = 0$, where φ runs through $\operatorname{Ann} U \subset V^*$. For example, the projectivized linear span of any two non-proportional vectors $a, b \in V$ is denoted $(ab) \subset \mathbb{P}(V)$ and called *a line*. It consists of ll points represented by the vectors $\lambda a + \mu b$, $\lambda, \mu \in \Bbbk$. Alternatively, it is described by the system of linear equations $\xi(x) = 0$, where ξ runs through the subspace $\operatorname{Ann}(a) \cap \operatorname{Ann}(b) \subset V^*$ or, equivalently, through an arbitrary basis of this subspace. The ratio $(\lambda : \mu)$ can be considered as the internal homogeneous coordinate of the point $\lambda a + \mu b$ on the projective line (ab) with respect to the basis a, b.

EXERCISE 1.6. Show that $\dim K \cap L \geqslant \dim K + \dim L - n$ for any two projective subspaces $K, L \subset \mathbb{P}_n$. In particular, $K \cap L \neq \varnothing$ soon $\dim K + \dim L \geqslant n$. For example, any two lines on $\mathbb{P}_2$ are intersecting.

EXAMPLE 1.3 (REAL AFFINE CONICS)
Consider the real projective plane $\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3)$ and the curve C defined by homogeneous equation

$$x_0^2 + x_1^2 = x_2^2. \tag{1-2}$$

[1] Note that the boundary of a Möbius tape is a circle as well as the boundary of a disc.

In the standard affine chart U_2, where $x_2 = 1$, in the default local affine coordinates $t_0 = x_0 / x_2$, $t_1 = x_1 / x_2$, the equation (1-2) turns to the equation of circle $t_0^2 + t_1^2 = 1$. In the chart U_1, where $x_1 = 1$, in the coordinates $t_0 = x_0 / x_1$, $t_2 = x_2 / x_1$, we get the hyperbola $t_2^2 - t_0^2 = 1$. In the «slanted» chart $U_{x_1+x_2}$, where $x_1 + x_2 = 1$, in the coordinates

$$s = x_0|_{U_{x_1+x_2}} = \frac{x_0}{x_1 + x_2}, \quad t = (x_2 - x_1)|_{U_{x_1+x_2}} = \frac{x_2 - x_1}{x_2 + x_1},$$

the equation (1-2) turns[1] to the equation of parabola $s^2 = t$. Thus, the affine ellipse, hyperbola, and parabola are just different pieces of the same projective curve C observed in several affine charts. The shape of C in an affine chart $U_\xi \subset \mathbb{P}_2$ is determined by the positional relationship between C and the infinite line $\ell_\infty = V(\xi)$ of the chart U_ξ. The curve C is looking as an ellipse, hyperbola, and parabola as soon ℓ_∞ does not intersect C, touches C at one point, and intersects C in two distinct points respectively, see. fig. 1◇5.

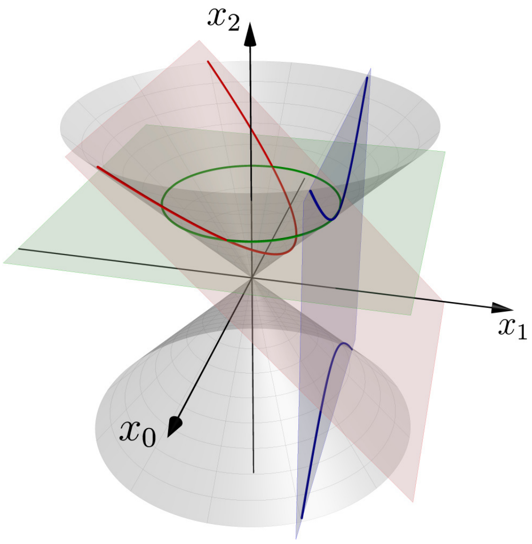


Fig. 1◇5. Real projective conic.

1.3.1 Projective closure of affine variety. The affine space $\mathbb{A}^n = \mathbb{A}(\Bbbk^n)$ with coordinates

$$(x_1, x_2, \dots, x_n)$$

can be considered as the standard affine chart U_0 in the projective space $\mathbb{P}_n = \mathbb{P}\left(\Bbbk^{n+1}\right)$ with homogeneous coordinates $(x_0 : x_1 : \dots : x_n)$. Every affine algebraic hypersurface $S = V(f) \subset \mathbb{A}^n$, where $f(x_1, x_2, \dots, x_n)$ is a (non-homogeneous) polynomial of degree d, admits the canonical extension to the projective hypersurface $\overline{S} = V(\overline{f}) \subset \mathbb{P}_n$ called the *projective closure* of S and defined by the homogeneous polynomial $\overline{f}(x_0, x_1, \dots, x_n) \in S^d V^*$ of the same degree d such that

$$\overline{f}(1, x_1, \dots, x_n) = f(x_1, x_2, \dots, x_n).$$

This polynomial is constructed as follows: write f as

$$f(x_1, x_2, \dots, x_n) = f_0 + f_1(x_1, x_2, \dots, x_n) + f_2(x_1, x_2, \dots, x_n) + \cdots + f_d(x_1, x_2, \dots, x_n)$$

[1]Move x_1^2 to the right hand side of (1-2) and divide the both sides by $(x_2 + x_1)^2$.

where every component f_i is homogeneous of degree i, and put

$$\overline{f}(x_0, x_1, \dots, x_n) = f_0 \cdot x_0^d + f_1(x_1, x_2, \dots, x_n) \cdot x_0^{d-1} + \cdots + f_d(x_1, x_2, \dots, x_n) .$$

Note that $\overline{S} \cap U_0 = S$ and the complement $\overline{S} \smallsetminus S = \overline{S} \cap U_0^{(\infty)}$ is cut out of $\overline{S}$ by the infinite hyperplane $x_0 = 0$ of the chart U_0. In terms of the standard homogeneous coordinates $(x_1 : x_2 : \cdots : x_n)$ on the infinite hyperplane, the intersection with $\overline{S}$ is described by the homogeneous equation

$$f_d(x_1, x_2, \dots, x_n) = 0$$

of degree d, that is, by the vanishing of top degree homogeneous component of the polynomial f describing S. Thus, the infinite points of $\overline{S}$ are nothing else than the *asymptotic directions* of affine hypersurface S.

For example, the projective closure of affine cubic curve $x_1 = x_2^3$ is the projective cubic $x_0^2 x_1 = x_2^3$. The latter has exactly one infinite point $p_\infty = (0 : 1 : 0)$. In the standard chart U_1, which covers this point, the curve looks like the semi-cubic parabola $x_0^2 = x_2^3$ with a cusp at p_∞.

1.3.2 Space of hypersurfaces. Since proportional polynomials define the same hypersurfaces $V(f) = V(\lambda f)$, the projective hypersurfaces of a fixed degree d can be viewed as the points of projective space $\mathcal{S}_d = \mathcal{S}_d(V) \overset{\text{def}}{=} \mathbb{P}(S^d V^*)$, which is called the *space of degree d hypersufaces* in $\mathbb{P}(V)$.

EXERCISE 1.7. Find $\dim \mathcal{S}_d(V)$ assuming that $\dim V = n + 1$.

Projective subspaces of $\mathcal{S}_d$ are called *linear systems* of hypersurfaces. For example, all degree d hypersurfaces passing through a given point $p \in \mathbb{P}(V)$ form a linear system of codimension one, i.e., a hyperplane in $\mathcal{S}_d$, because the equation $f(p) = 0$ is linear in $f \in S^d V^*$. Every hypersurface laying in a linear system spanned by $V(f_1)$, $V(f_2)$, … , $V(f_m)$, is given by equation of the form

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m = 0 , \quad \text{where } \lambda_1, \lambda_2, \dots, \lambda_m \in \Bbbk .$$

In particular, any such a hypersurface contains the intersection locus $V(f_1) \cap V(f_2) \cap \ldots \cap V(f_m)$. The points of this intersection are called the *base points* of the linear system. Traditionally, linear systems of dimensions 1, 2, 3 are called *pencils*, *nets*, and *webs* respectively.

EXERCISE 1.8. Show that each pencil of hypersurfaces contains a hypersurface passing through an arbitrarily prescribed point.

CAUTION 1.1. It should be kept in mind that if the ground field is not algebraically closed, then some polynomials of degree d may determine nothing geometrically reminiscent of a hypersurface of degree d. For example, the equation $x_0^2 + x_1^2 = 0$ over $\mathbb{R}$ describes the empty set $\varnothing$ on the projective line $\mathbb{P}_1$, and the one point set $(0 : 0 : 1)$ in the projective plane $\mathbb{P}_2$. Even over an algebraically closed field, some distinct points $f \neq g$ in $\mathbb{P}(S^d V^*)$ produce the same zero set $V(f) = V(g)$ in $\mathbb{P}(V)$. For example, the non-proportional polynomials $x_0^2 x_1$ and $x_0 x_1^2$ define the same two-point set $\{(0 : 1), (1 : 0)\}$ on $\mathbb{P}_1$. We postpone the discussion of geometric concepts avoiding such problems up to **??**.

1.3.3 Working example: unordered collections of points on the line. Let $U = \Bbbk^2$ with the standard coordinates x_0, x_1. Every set of d not necessary distinct points $p_1, p_2, \dots, p_d \in \mathbb{P}_1 = \mathbb{P}(U)$ is the zero set of homogeneous polynomial of degree d

$$f(x_0, x_1) = \prod_{\nu=1}^{d} \det(x, p_\nu) = \prod_{\nu=1}^{d} (p_{\nu,1} x_0 - p_{\nu,0} x_1) , \quad \text{where } p_\nu = (p_{\nu,0} : p_{\nu,1}) , \tag{1-3}$$

which is predicted by the set uniquely up to a scalar factor. We say that the points p_i are the *roots* of f. Each non-zero homogeneous polynomial of degree d has at most d distinct roots on $\mathbb{P}_1$. If the ground field $\Bbbk$ is algebraically closed, the number of roots[1] equals d, and sending a collection of points $p_1, p_2, \dots, p_d$ to the polynomial (1-3) establishes the bijection between the non-ordered d-typles of points on $\mathbb{P}_1$ and the points of projective space $\mathbb{P}(S^d U^*)$.

For an arbitrary field $\Bbbk$, those collections where all d points coincide form a curve

$$C_d \subset \mathbb{P}_d = \mathbb{P}(S^d U^*)$$

called the *Veronese curve*[2] of degree d. It coincides with the image of the *Veronese embedding*

$$v_d : \mathbb{P}_1^\times = \mathbb{P}(U^*) \hookrightarrow \mathbb{P}_d = \mathbb{P}\left(S^d U^*\right), \quad \varphi \mapsto \varphi^d, \tag{1-4}$$

that takes a linear form $\varphi \in U^*$, whose zero set consists of one point $p = \operatorname{Ann}\varphi \in \mathbb{P}_1 = \mathbb{P}(U)$, to the d th power $\varphi^d \in S^d(U^*)$, whose zero set is the d-tiple point p.

Now assume that $\operatorname{char}\Bbbk = 0$, write polynomials $\varphi \in U^*$ and $f \in S^d(U^*)$ in the form[3]

$$\varphi(x) = \alpha_0 x_0 + \alpha_1 x_1, \quad f(x) = \sum_\nu a_\nu \cdot \binom{d}{\nu} x_0^{d-\nu} x_1^\nu,$$

and use $\alpha = (\alpha_0 : \alpha_1)$ and $a = (a_0 : a_1 : \dots : a_d)$ as homogeneous coordinates in the spaces $\mathbb{P}_1^\times = \mathbb{P}(U^*)$ and $\mathbb{P}_d = \mathbb{P}(S^d U^*)$ respectively. Then we get the following parameterization of the Veronese curve by the points of $\mathbb{P}_1^\times$:

$$(\alpha_0 : \alpha_1) \mapsto (a_0 : a_1 : \dots : a_d) = \left(\alpha_0^d : \alpha_0^{d-1}\alpha_1 : \alpha_0^{d-2}\alpha_1^2 : \dots : \alpha_1^d\right). \tag{1-5}$$

It shows that C_d consists of all those $(a_0 : a_1 : \dots : a_d) \in \mathbb{P}_d$ that form a geometric progression, i.e., such that the rows of matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{d-2} & a_{d-1} \\ a_1 & a_2 & a_3 & \dots & a_{d-1} & a_d \end{pmatrix}$$

are proportional. The condition $\operatorname{rk} A = 1$ is equivalent to the system of homogeneous quadratic equations $a_i a_{j+1} = a_{i+1} a_j$ saying that all 2×2-minors of A vanish. Thus, $C_d \subset \mathbb{P}_d$ is an algebraic projective variety rationally parameterized by the points of projective line. The intersection of C_d with an arbitrary hyperplane in $\mathbb{P}_d$ given by linear equation $A_0 a_0 + A_1 a_1 + \dots + A_d a_d = 0$ consists of the Veronese-images of roots $(\alpha_0 : \alpha_1) \in \mathbb{P}_1$ of homogeneous polynomial $\sum_\nu A_\nu \cdot \alpha_0^{d-\nu}\alpha_1^\nu$ of degree d. Since it has at most d roots, any $d+1$ distinct points on the Veronese curve do not lie in a hyperplane. This implies that for $2 \leqslant m \leqslant d+1$, any m distinct points of C_d span a subspace of dimension $m-1$ and do not lie in a subspace of dimension $(m-2)$.

EXERCISE 1.9. Make sure that this fails when $\operatorname{char}\Bbbk$ is positive and divides d.

If $\Bbbk$ is algebraically closed, C_d intersects any hyperplane in precisely d points (some of which may coincide). By this reason we say that C_d has degree d.

[1]Counted with *multiplicities*, where the multiplicity of a root p is defined as the maximal integer k such that $\det^k(x, p)$ divides f in $\Bbbk[x_0, x_1]$.

[2]It has several other names: *rational normal curve*, *twisted rational curve of degree d* etc

[3]Note that for $\operatorname{char}\Bbbk > 0$, the binomial coefficients $\binom{d}{\nu}$ may vanish and can not be factored out the coefficients of f.

EXAMPLE 1.4 (VERONESE CONIC)
The Veronese conic $C_2 \subset \mathbb{P}_2$ consists of quadratic trinomials $a_0x_0^2+2a_1x_0x_1+a_2x_1^2$ that are perfect squares of linear forms. It is given by the equation $D/4 = -\det\left(\begin{smallmatrix} a_0 & a_1 \\ a_1 & a_2 \end{smallmatrix}\right) = a_1^2 - a_0a_2 = 0$ and comes with the rational parametrization $a_0 = \alpha_0^2$, $a_1 = \alpha_0\alpha_1$, $a_2 = \alpha_1^2$.

1.4 Complementary subspaces and projections. Projective subspaces $K = \mathbb{P}(U)$ and $L = \mathbb{P}(W)$ in $\mathbb{P}_n = \mathbb{P}(V)$ are called *complementary*, if $K \cap L = \varnothing$ and $\dim K + \dim L = n-1$. For example, any two non-intersecting lines in $\mathbb{P}_3$ are complementary. In terms of the linear algebra, the complementarity of K, L means that the vector subspaces $U, W \subset V$ have zero intersection $U \cap V = 0$ and

$$\dim U + \dim W = \dim K + 1 + \dim L + 1 = n + 1 = \dim V\,,$$

i.e., $V = U \oplus W$. In this case every vector $v \in V$ has a unique decomposition $v = u + w$, where $u \in U$, $w \in W$. In particular, $v \notin U \cup W$ if and only if the both components u, w are non zero. Geometrically, this means that every point $p \notin K \sqcup L$ lies on a unique line intersecting the both subspaces K, L.

EXERCISE 1.10. Make it sure.

For a pair of complementary subspaces $K, L \subset \mathbb{P}_n$, the projection $\pi_L^K : (\mathbb{P}_n \smallsetminus K) \to L$ from K onto L acts identically on L and sends every point $p \notin K \sqcup L$ to the unique point $b \in L$ such that the line (pb) intersects K. In homogeneous coordinates $(x_0 : x_1 : \ \dots\ : x_n)$ such that $(x_0 : x_1 : \ \dots\ : x_m)$ are the coordinates in K and $(x_{m+1} : x_{m+2} : \ \dots\ : x_n)$ are the coordinates in L, the projection π_L^K just removes the first $m+1$ coordinates x_ν, $0 \leqslant \nu \leqslant m$.

EXAMPLE 1.5 (PROJECTING A CONIC TO A LINE)
Let $C, L \subset \mathbb{P}_2$ be the conic and line given by equations[1] $x_0^2 + x_1^2 = x_2^2$ and $x_0 = 0$ respectively. Consider the projection $\pi_L^p : C \to L$ of C to L from $p = (1 : 0 : 1) \in C$ and extend it to p by sending p to $(0 : 1 : 0) \in L$, the intersection point of L with the tangent line to C at p. In the standard affine chart U_2 this looks as on fig. 1⋄6. Clearly, π_L^p provides a bijection between L and C. This bijection is *birational*: the homogeneous coordinates of the corresponding points

$$q = (q_0 : q_1 : q_2) \in C$$
$$t = (0 : t_1 : t_2) = \pi_L^p(q) \in L$$

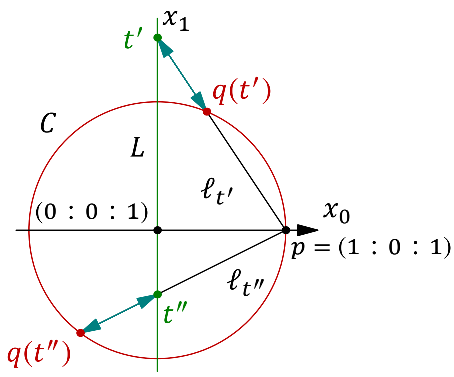


Fig. 1⋄6. Projecting a conic to a line.

are *rational* algebraic functions of each other:

$$(t_1 : t_2) = (q_1 : q_2 - q_0), \quad (q_0 : q_1 : q_2) = (t_1^2 - t_2^2 : 2t_1t_2 : t_1^2 + t_2^2)$$

EXERCISE 1.11. Check these formulas and use the second of them to list all integer solutions of the Pythagor equation $a^2 + a^2 = c^2$ up to common integer factor.

The invertible linear change of homogeneous coordinates by formulas

$$\begin{cases} a_0 = x_2 + x_0 \\ a_1 = x_1 \\ a_2 = x_2 - x_0 \end{cases} \qquad \begin{cases} x_0 = (a_0 - a_2)/2 \\ x_1 = a_1 \\ x_0 = (a_0 + a_2)/2 \end{cases}$$

[1] It is the same as in Example 1.3 on p. 7 above.

transforms C to the Veronese conic $a_1^2 = a_0a_2$ from Example 1.4 on p. 11 and turns the above parameterization to the standard parameterization of Veronese conic.

1.5 Linear projective transformations. Any linear isomorphism of vector spaces $F : U \xrightarrow{\sim} W$ produces well defined bijection $\overline{F} : \mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$ called *a linear projective isomorphism*.

EXERCISE 1.12. Given two hyperplanes $L_1, L_2 \subset \mathbb{P}_n = \mathbb{P}(V)$ and a point $p \notin L_1 \cup L_2$, verify that a projection from p to L_2 induces a linear projective isomorphism $\gamma_p : L_1 \xrightarrow{\sim} L_2$.

THEOREM 1.1
For any two vector spaces U, W of the same dimension $n+1$ and two ordered collections of $n+2$ points $p_0, p_1, \dots, p_{n+1} \in \mathbb{P}(U)$, $q_0, q_1, \dots, q_{n+1} \in \mathbb{P}(W)$ such that no $n+1$ points of each collection lie in a hyperplane, there exists a unique up scalar factor linear isomorphism of vector spaces $F : U \xrightarrow{\sim} W$ such that $\overline{F}(p_i) = q_i$ for all i.

PROOF. Fix some vectors u_i, w_i representing the points p_i, q_i and chose the vectors $u_0, u_1, \dots, u_n$ and $w_0, w_1, \dots, w_n$ as the bases in U and W. The condition $\overline{F}(p_i) = q_i$ means that $F(u_i) = \lambda_i w_i$ for some non zero $\lambda_i \in \Bbbk$. Thus, the matrix of F in chosen bases is diagonal with $\lambda_0, \lambda_1, \dots, \lambda_n$ on the diagonal. Further, all coordinates x_i in the expansion $u_{n+1} = x_0u_0 + x_1u_1 + \cdots + x_nu_n$ are non zero, because vanishing of x_k forces $n+1$ points p_j with $j \neq k$ lie in the hyperplane $x_k = 0$. The same holds for the expansion $w_{n+1} = y_0w_0 + y_1w_1 + \cdots + y_nw_n$, certainly. The condition $F(u_{n+1}) = \lambda_{n+1}w_{n+1}$ implies that $\lambda_ix_i = \lambda_{n+1}y_i$ for all $0 \leqslant i \leqslant n$. Therefore the diagonal elements $\lambda_i = \lambda_{n+1} \cdot y_i / x_i$, $0 \leqslant i \leqslant n$, are uniquely determined by $\overline{F}$ up to non zero scalar factor λ_{n+1}. □

COROLLARY 1.1
Two linear isomorphisms of vector spaces $F, G : U \xrightarrow{\sim} W$ produce the same linear projective isomorphism $\overline{F} = \overline{G} : \mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$ if and only if $F = \lambda G$ for some non zero $\lambda \in \Bbbk$. □

EXAMPLE 1.6 (AUTOMORPHISMS OF QUADRANGLE)
A figure formed by 4 points $p_1, p_2, p_3, p_4 \in \mathbb{P}_2$ any 3 of which are non-collinear and 6 lines joining the points like on fig. 1◇7 is called *a quadrangle*. The intersection points of its opposite sides:

$$q_1 = (p_1p_2) \cap (p_3p_4)$$
$$q_2 = (p_1p_3) \cap (p_2p_4)$$
$$q_3 = (p_1p_4) \cap (p_2p_3)$$

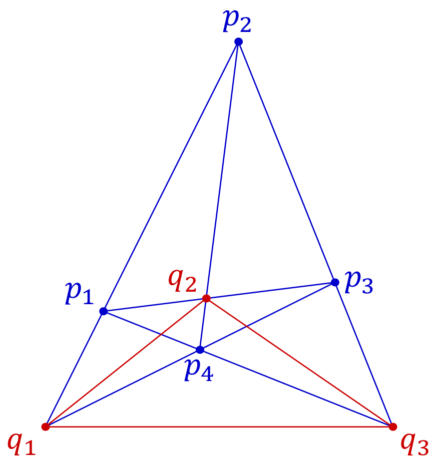


Fig. 1◇7. Quadrangle and associated triangle.

and 3 lines joining them form the *associated triangle* of the quadrangle. Every linear projective automorphism of $\mathbb{P}_2$ sending the quadrangle to itself permutes its vertexes, and every permutation of the vertexes is uniquely extended to a linear projective automorphism of $\mathbb{P}_2$ by Theorem 1.1. Hence, the group of all linear projective automorphism of $\mathbb{P}_2$ sending the quadrangle to itself is naturally identified with the symmetric group S_4. Every transformation from this group permutes the vertexes of associated triangle. This leads to the surjective homomorphism of groups $S_4 \twoheadrightarrow S_3$. Its kernel is the

Klein's normal subgroup

$$V_4 = \big\{(1,2,3,4),\,(2,1,4,3),\,(3,4,1,2),\,(4,3,2,1)\big\} \lhd S_4$$

formed by the identity permutation and 3 pairs of independent transpositions. The transpositions (12), (13), (23) and 3-cycles (123), (132) from the group S_4 are mapped to the same transpositions (12), (13), (23) and 3-cycles (123), (132) from the group S_3, see fig. 1◇7.

1.5.1 Projective linear group. Linear projective automorphisms of $\mathbb{P}(V)$ form a group called the *projective linear group* of V and denoted $\mathrm{PGL}(V)$. It follows from Theorem 1.1 that this group is isomorphic to the quotient of linear group $\mathrm{GL}(V)$ by the subgroup of scalar dilatations. A choice of basis in V identifies $\mathrm{GL}(V)$ with the group $\mathrm{GL}_{n+1}(\Bbbk)$ of non-degenerated square matrices. Then $\mathrm{PGL}(V)$ is identified with group $\mathrm{PGL}_{n+1}(\Bbbk)$ of the same matrices considered up to proportionality. Such a matrix A acts on a point $x = (x_0 : x_1 : \dots : x_n) \in \mathbb{P}_n$ via left multiplication of the coordinate column: $x \mapsto (Ax^t)^t = xA^t$, where M^t means the transposed M.

EXAMPLE 1.7 (LINEAR FRACTIONAL TRANSFORMATIONS OF LINE)
The group $\mathrm{PGL}_2(\Bbbk)$ consists of non-degenerated 2×2-matrices $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ with $\alpha\delta - \beta\gamma \neq 0$ considered up to a constant factor. Such a matrix acts on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ by the rule

$$(x_0 : x_1) \mapsto (ax_0 + \beta x_1 : \gamma x_0 + \delta x_1).$$

In the standard affine chart $U_1 \simeq \mathbb{A}^1$ this action performs the linear fractional transformation of the local coordinate $t = x_0/x_1$ by the rule $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$. Clearly, this transformation is not changed under rescaling of the matrix A. For any triple of distinct points q, r, s, there is a unique linear fractional map sending them to ∞, 0, 1 respectively. Indeed, this map is forced to take

$$t \mapsto \frac{t-r}{t-q}\cdot\frac{s-r}{s-q}\,. \tag{1-6}$$

1.5.2 Cross-ratio. Given two points $a = (a_0 : a_1)$, $b = (b_0 : b_1)$ on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$, the difference of their affine coordinates in the standard chart U_1 is expressed trough the determinant of their homogeneous coordinates by the formula

$$a - b = \frac{a_0}{a_1} - \frac{b_0}{b_1} = \frac{a_0b_1 - a_1b_0}{a_1b_1} = \frac{\det(a,b)}{a_1b_1}\,.$$

For an ordered quadruple of distinct points $p_1, p_2, p_3, p_4 \in \mathbb{P}_1$, the quantity

$$[p_1,p_2,p_3,p_4] \overset{\text{def}}{=} \frac{(p_1-p_3)(p_2-p_4)}{(p_1-p_4)(p_2-p_3)} = \frac{\det(p_1,p_3)\cdot\det(p_2,p_4)}{\det(p_1,p_4)\cdot\det(p_2,p_3)} \tag{1-7}$$

is called the *cross-ratio* of the quadruple p_1, p_2, p_3, p_4. It follows from (1-6) that $[p_1,p_2,p_3,p_4]$ equals the affine coordinate of image of the point p_4 under the linear projective isomorphism sending p_1, p_2, p_3 to ∞, 0, 1 respectively. It can take any value except for ∞, 0, 1.

EXERCISE 1.13. Prove that two ordered quadruples of distinct points on $\mathbb{P}_1$ can be transformed one to the other by a linear projective automorphism if and only if they have equal cross-ratios.

Since an invertible linear change of homogeneous coordinates is nothing but a linear projective automorphism, the right hand side of (1-7) does not depend on the choice of coordinates on $\mathbb{P}_1$. This forces the middle part of (1-7) to depend neither on the choice of affine chart containing the points[1] nor on the choice of local affine coordinate within the chart. The symmetric group S_4 acts on every given quadruple of points by permutations. It is clear from (1-7) that the Klein subgroup $V_4 \subset S_4$ preserves the cross-ratio: $[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_1, p_2] = [p_4, p_3, p_2, p_1]$.

EXERCISE 1.14. Check that the values of cross-ratio appearing under the action of V_4-cosets of identity, transpositions (12), (13), (23), and 3-cycles (123), (132) are related as follows:

$$\begin{aligned}
[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_1, p_2] = [p_4, p_3, p_2, p_1] &= \vartheta \\
[p_2, p_1, p_3, p_4] = [p_1, p_2, p_4, p_3] = [p_3, p_4, p_2, p_1] = [p_4, p_3, p_1, p_2] &= 1/\vartheta \\
[p_3, p_2, p_1, p_4] = [p_2, p_3, p_4, p_1] = [p_1, p_4, p_3, p_2] = [p_4, p_1, p_2, p_3] &= \vartheta/(\vartheta-1) \\
[p_1, p_3, p_2, p_4] = [p_3, p_1, p_4, p_2] = [p_2, p_4, p_1, p_3] = [p_4, p_2, p_3, p_1] &= 1-\vartheta \\
[p_2, p_3, p_1, p_4] = [p_3, p_2, p_4, p_1] = [p_1, p_4, p_2, p_3] = [p_4, p_1, p_3, p_2] &= (\vartheta-1)/\vartheta \\
[p_3, p_1, p_2, p_4] = [p_1, p_3, p_4, p_2] = [p_2, p_4, p_3, p_1] = [p_4, p_2, p_1, p_3] &= 1/(1-\vartheta)\,.
\end{aligned} \tag{1-8}$$

These formulas show that there are three special values[2] $[p_1, p_2, p_3, p_4] = -1,\ 2,\ 1/2$ preserved, respectively, by the transpositions (12), (13), (23) and cyclically permuted by the 3-cycles. Similarly, there are two special values preserved by the 3-cycles and interchanged by the transpositions. They satisfy the equivalent quadratic equations[3] $\vartheta = (\vartheta-1)/\vartheta \Leftrightarrow \vartheta^2 - \vartheta + 1 = 0 \Leftrightarrow \vartheta = 1/(1-\vartheta)$.

The five just listed values of $[p_1, p_2, p_3, p_4]$ are called *special*. The quadruples of points with such cross-ratios are also called *special*. The permutations of points in a non-special quadruple lead to 6 distinct values of the cross-ratio. For a special quadruple we get either 3 or 2 distinct values.

1.5.3 Harmomic pairs of points. A special quadruple of points $a, b, c, d \in \mathbb{P}_1$ with $[a, b, c, d] = -1$ is called *harmonic*. Geometrically, this means that b is the middle point of $[c, d]$ in the affine chart with the infinity at a. Algebraically, the harmonicity means that the cross-ratio is changed neither by the transposition (12), nor by the transposition (34), and each of these two properties forces the quadruple to be harmonic. Since the order preserving exchange of a, b with c, d keeps the cross-ratio fixed, the harmonicity is a symmetric binary relation on the set of non-ordered pairs of distinct points in $\mathbb{P}_1$.

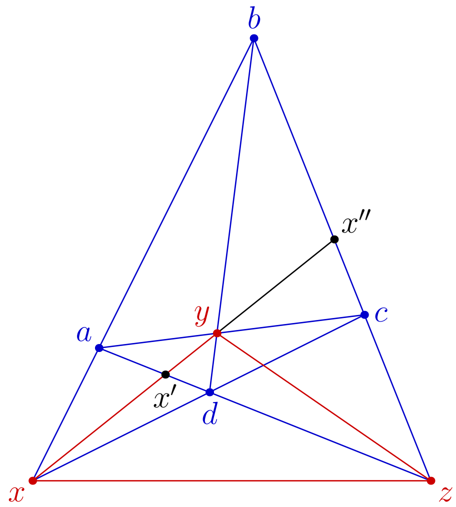


Fig. 1⋄8. Harmonic pairs of sides.

PROPOSITION 1.2 (HARMONICITY IN QUADRANGLE)
For any quadrangle a, b, c, d on $\mathbb{P}_2$ and its associated triangle x, y, z, the sides of quadrangle are harmonic to the sides of triangle in the pencils of lines passing through the vertexes of triangle.

PROOF. We verify the proposition at the vertex x. The pencil of lines passing through x is parameterized by the points of line (ad) by sending a point $p \in (ad)$ to the line (xp). We have to

[1]Algebraically, this means that all four values $p_1, p_2, p_3, p_4 \in \Bbbk$ are finite.
[2]They satisfy the equations $\vartheta = 1/\vartheta$, $\vartheta = \vartheta/(\vartheta-1)$, and $\vartheta = 1-\vartheta$.
[3]That is, coincide with two different from -1 cubic roots of one as soon those exist in $\Bbbk$.

check that $[a, d, z, x'] = -1$, see fig. 1⋄8. Since the central projections from x and y preserve the cross-ratios, $[a, d, z, x'] = [b, c, z, x''] = [d, a, z, x']$. Since the transposition in the first pair of points does not change the cross-ratio, the latter equals -1. □

§2 Projective Quadrics

2.1 Quadratic forms and quadrics. We assume on default in §2 that char $\Bbbk \neq 2$. Projective hypersurfaces of degree 2 are called *projective quadrics*. Given a non-zero quadratic form $q \in S^2V^*$, we write $Q = V(q) \subset \mathbb{P}(V)$ for the quadric provided by the zero set of q.

2.1.1 The Gram matrix. If char $\Bbbk \neq 2$, then every quadratic form $q \in S^2V^*$ on $V = \Bbbk^{n+1}$ can be written as $q(x) = \sum_{i,j} a_{ij}\, x_i x_j = xAx^t$, where $x = (x_0, x_1, \dots, x_n)$ is the coordinate row, x^t is the transposed column of coordinates, and $A = \left(a_{ij}\right) \in \mathrm{Mat}_{n+1}(\Bbbk)$ is a *symmetric* square matrix. Every non-diagonal element $a_{ij} = a_{ji}$ of A equals the half[1] of coefficient of monomial $x_i x_j$ in the reduced expansion for q. The matrix A is called *the Gram matrix* of q in the chosen basis of V.

In other words, for any quadratic polynomial q on V, there exists a unique symmetric bilinear form $\widetilde{q} :\ V \times V \to \Bbbk$ such that $q(v) = \widetilde{q}(v, v)$ for all $v \in V$. In coordinates,

$$\widetilde{q}(x, y) = \sum a_{ij} x_i y_j = xAy^t = \frac{1}{2} \sum y_i \frac{\partial q(x)}{\partial x_i} \,. \tag{2-1}$$

In coordinate-free terms, $\widetilde{q}(x, y) = \frac{1}{2}\big(q(x + y) - q(x) - q(y)\big) = \frac{1}{4}\big(q(x + y) - q(x - y)\big)$.

Exercise 2.1. Check this.

The symmetric bilinear form $\widetilde{q}$ is called *the polarization* of quadratic form q. It can be thought of as an inner product on V, possibly degenerated. The elements of Gram matrix equal the inner products of basic vectors: $a_{ij} = \widetilde{q}(e_i, e_j)$. In the matrix notations, $A = e^t \cdot e$, where $e = (e_0, e_1, \dots, e_n)$ is the row of basic vectors in V, e^t is the transposed column of basic vectors, and $u \cdot w \overset{\text{def}}{=} \widetilde{q}(u, w) \in \Bbbk$ for $u, w \in V$. When we pass to another basis $e' = eC$, where $C \in \mathrm{GL}_{n+1}(\Bbbk)$, the Gram matrix A of e is related with the Gram matrix A' of e' as $A' = C^tAC$, because $(e')^t \cdot e' = C^t e^t \cdot eC$.

2.1.2 The Gram dterminant. Since $\det A' = \det A \cdot \det^2 C$, the determinant of Gram matrix does not depend on the choice of basis up to multiplication by non zero squares from $\Bbbk$. We write $\det q \in \Bbbk/\Bbbk^{*2}$ for the class of $\det A$ modulo multiplication by non zero squares, and call it *the Gram determinant* of quadratic form $q \in S^2V^*$. The form q and quadric $Q = V(q)$ are called *smooth* or *non-singular*, if $\det q \neq 0$. Otherwise they are called *singular* or *degenerated*.

2.1.3 The rank. Since the rank of matrix is not changed under multiplications of the matrix by non-degenerated matrices, the rank of Gram matrix does not depend on the choice of basis as well. It is called *the rang* of quadretic form q and quadric $Q = V(q)$, and denoted by $\operatorname{rk} q = \operatorname{rk} Q \overset{\text{def}}{=} \operatorname{rk} A$.

Proposition 2.1 (Lagrange's theorem)
For any quadratic form q there exists a basis where the Gram matrix of q is diagonal.

Proof. Induction on $\dim V$. If $q \equiv 0$ or $\dim V = 1$, then the Gram matrix is diagonal. If $\dim V \geqslant 2$ and $q(e) = \widetilde{q}(e, e) \neq 0$ for some $e \in V$, we put $e_1 = e$ to be the first vector of desired basis. Every vector $v \in V$ admits a unique decomposition $v = \lambda e + u$, where $\lambda \in \Bbbk$ and $u \in v^\perp = \{\, w \in V \mid \widetilde{q}(v, w) = 0 \,\}$. Indeed, the orthogonality of v and $v - \lambda e$ forces $\lambda = \widetilde{q}(e, v)/\widetilde{q}(e, e)$, then $u = v - (\widetilde{q}(e, v)/\widetilde{q}(e, e)) \cdot e$.

Exercise 2.2. Verify that $v - \big(\widetilde{q}(e, v)/\widetilde{q}(e, e)\big) \cdot e \in e^\perp$.

Thus, we have the orthogonal decomposition $V = \Bbbk \cdot e \oplus e^\perp$. By induction, there exists a basis $e_2, \dots, e_n$ in $e^\perp$ with diagonal Gram matrix. Hence, $e_1, e_2, \dots, e_n$ is a required basis for V. □

[1]Note that if char $\Bbbk = 2$, such the matrix A does not always exists.

Corollary 2.1
Every quadratic form q over an algebraically closed field turns to the sum of squares

$$q(x) = x_0^2 + x_1^2 + \cdots + x_k^2 , \quad \text{where } k + 1 = \operatorname{rk} q ,$$

in appropriate coordinates on V.

Proof. Pass to a basis $e_0, e_1, \dots, e_n$ in which the Gram matrix is diagonal, renumber the vectors e_i in order to have $q(e_i) \neq 0$ exactly for $1 \leqslant i \leqslant k$, then multiply all these e_i by $1/\sqrt{q(e_i)} \in \Bbbk$. □

Example 2.1 (quadrics on $\mathbb{P}_1$)
It follows from Proposition 2.1 that the equation of any quadric $Q \subset \mathbb{P}_1$ can be written in appropriate coordinates on $\mathbb{P}_1$ either as $x_0^2 = 0$ or as $x_0^2 + ax_1^2 = 0$, where $a \neq 0$. In the first case, Q is singular, $\operatorname{rk} Q = 1$, and the equation of Q is the squared linear equation of the point $(0 : 1)$. By this reason, such a quadric is called a *double point*. In the second case, $\operatorname{rk} Q = 2$, the quadric is smooth, and its Gram determinant equals a up to multiplication by non-zero squares. If $-a \in \Bbbk$ is not a square, then the equation $(x_0/x_1)^2 = -a$ has no solutions, and the quadric is empty. If $-a = \delta^2$ for some $\delta \in \Bbbk$, then $x_0^2 + ax_1^2 = (x_0 - \delta x_1)(x_0 + \delta x_1)$ has two distinct roots $(\pm\delta : 1) \in \mathbb{P}_1$. Thus, the geometry of quadric $Q = V(q) \subset \mathbb{P}_1$ is completely determined by the Gram determinant $\det q \in \Bbbk/(\Bbbk^*)^2$. If $\det q = 0$, then the quadric is a double point. If $-\det q = 1$, that is, $-\det A \in (\Bbbk^*)^2$ is a non zero square, then the quadric consists of two distinct points. If $-\det q \neq 1$, that is, $-\det A \in \Bbbk$ is not a square, then the quadric is empty. Note that the latter case never appears over an algebraically closed field $\Bbbk$.

2.2 Tangent lines. It follows from Example 2.1 that there are precisely 4 different positional relationships between a quadric Q and a line ℓ in $\mathbb{P}_n$: either $\ell \subset Q$, or $\ell \cap Q$ is a double point, or $\ell \cap Q$ is a pair of distinct points, or $\ell \cap Q = \varnothing$, and the latter case never appears over an algebraically closed field.

Definition 2.1 (tangent space of quadric)
A line ℓ is called *tangent* to a quadric Q at a point $p \in Q$, if either $p \in \ell \subset Q$ or $Q \cap \ell$ is the double point p. In these cases we say that ℓ *touches* Q at p. The union of all tangent lines touching Q at a given point $p \in Q$ is called *the tangent space* to Q at p and denoted by T_pQ.

Proposition 2.2
A line (ab) touches a quadric $Q = V(q)$ at the point $a \in Q$ if and only if $\tilde{q}(a, b) = 0$.

Proof. The Gram matrix of restriction $q|_{(a,b)}$ in the basis a, b of line (ab) is

$$\begin{pmatrix} \tilde{q}(a,a) & \tilde{q}(a,b) \\ \tilde{q}(a,b) & \tilde{q}(b,b) \end{pmatrix} .$$

Since $\tilde{q}(a, a) = q(a) = 0$ by assumption, the Gram determinant $\det q|_{(a,b)} = \tilde{q}(a, b)^2$. It vanishes if and only if $\tilde{q}(a, b) = 0$. □

Corollary 2.2 (apparent contour of quadric)
For any point $p \notin Q$, the *apparent contour* of Q viewed from p, i.e., the set of all points $a \in Q$ such that the line (pa) touches Q at a, is cut out Q by the hyperplane $\Pi_p \stackrel{\text{def}}{=} \{x \in \mathbb{P}_n \mid \tilde{q}(p, x) = 0\}$.

Proof. Since $\tilde{q}(p, p) = q(p) \neq 0$, the equation $\tilde{q}(p, x) = 0$ is a non-trivial linear homogeneous equation on x. Thus, $\Pi_p \subset \mathbb{P}_n$ is a hyperplane, and $Q \cap \Pi$ coincides with the apparent contour of Q viewed from p by Proposition 2.2. □

2.2.1 Smooth and singular points. Associated with a quadratic form $q \in S^2V^*$ is the linear mapping

$$\widehat{q} : V \to V^*, \quad v \mapsto \widetilde{q}(\,*\,, v)\,, \tag{2-2}$$

sending a vector $v \in V$ to the linear form $\widehat{q}(v): V \to \Bbbk$, $w \mapsto \widetilde{q}(w, v)$. The map (2-2) is called *the correlation* of quadratic form q.

EXERCISE 2.3. Convince yourself that the matrix of linear map (2-2) written in dual bases e, x of V and V^* coincides with the Gram matrix of q in the basis e.

This shows once more, that the rank $\operatorname{rk} A = \dim V - \dim\ker\widehat{q}$ does not depend on a choice of basis. The vector space $\ker(q) \stackrel{\text{def}}{=} \ker\widehat{q} = \{\, v \in V \mid \widetilde{q}(w, v) = 0 \;\; \forall\, w \in V \,\}$ is called *the kernel* of quadratic form q. The projectivization of the kernel is denoted

$$\operatorname{Sing} Q \stackrel{\text{def}}{=} \mathbb{P}(\ker q) = \{p \in \mathbb{P}(V) \mid \forall\, u \in V \;\widehat{q}(p, u) = 0\}$$

and called *the vertex space* or *the singular locus* of quadric $Q = V(q) \subset \mathbb{P}_n$. The points of $\operatorname{Sing} Q$ are called *singular*. All points of the complement $Q \smallsetminus \operatorname{Sing} Q$ are called *smooth*. Thus, a point $p \in Q \subset \mathbb{P}(V)$ is smooth if and only if the tangent space $T_pQ = \{x \in \mathbb{P}_n \mid \widetilde{q}(p, x) = 0\}$ is a hyperplane in $\mathbb{P}_n$. Conversely, a point $p \in Q \subset \mathbb{P}(V)$ is singular if and only if the tangent space $T_pQ = \mathbb{P}(V)$ is the whole space, that is, any line passing through a either lies on Q or does not intersect Q anywhere besides a.

EXERCISE 2.4. Convince yourself that the singularity of a point $p \in Q \subset \mathbb{P}_n$ means that

$$\frac{\partial q}{\partial x_i}(p) = 0 \quad \text{for all } 0 \leqslant i \leqslant n.$$

Note that a quadric is smooth in the sense of n° 2.1.2 if and only if it has no singular points.

LEMMA 2.1
If a quadric $Q \subset \mathbb{P}_n$ has a smooth point $a \in Q$, then Q is not contained in a hyperplane.

PROOF. For $n = 1$, this follows from Example 2.1. Consider $n \geqslant 2$. If Q lies inside a hyperplane H, then every line $\ell \not\subset H$ passing through a intersects Q only in a and therefore is tangent to Q at a. Hence, $\mathbb{P}_n = H \cup T_pQ$. This contradicts to Exercise 2.5 below. □

EXERCISE 2.5. Show that the projective space over a field of characteristic $\neq 2$ is not a union of two hyperplanes.

THEOREM 2.1
For any quadric $Q \subset \mathbb{P}(V)$ and projective subspace $L \subset \mathbb{P}(V)$ complementary to $\operatorname{Sing} Q$, the intersection $Q' = L \cap Q$ is a smooth quadric in L, and Q is the linear join[1] of Q' and $\operatorname{Sing} Q$.

PROOF. Let $L = \mathbb{P}(U)$. Then $V = \ker q \oplus U$. Assume that there exists a vector $u \in U$ such that $\widetilde{q}(u, u') = 0$ for all $u' \in U$. Since $\widetilde{q}(u, w) = 0$ for all $w \in \ker q$ as well, the equality $\widetilde{q}(u, v) = 0$ holds for all $v \in V$. Hence, $u \in \ker q \cap U = 0$. That is, Q' is smooth. Every line ℓ that intersects $\operatorname{Sing} Q$ but is not contained in $\operatorname{Sing} Q$ does intersect L and either is contained in Q or does not intersect Q anywhere besides the point $\ell \cap \operatorname{Sing} Q$. This forces Q to be the union of lines (sp) such that $s \in \operatorname{Sing} Q$, $p \in L \cap Q$. □

[1] For sets $X, Y \subset \mathbb{P}_n$, their *linear join* is the union of all lines (xy) such that $x \in X$, $y \in Y$.

2.3 Duality. Projective spaces $\mathbb{P}_n = \mathbb{P}(V)$, $\mathbb{P}_n^\times \overset{\text{def}}{=} \mathbb{P}(V^*)$, obtained from dual vector spaces V, V^*, are called *dual*. Geometrically, $\mathbb{P}_n^\times$ is the space of hyperplanes in $\mathbb{P}_n$, and vice versa. The linear equation $\langle\, \xi\,,\, v\,\rangle = 0$, being considered as an equation on $v \in V$ for a fixed $\xi \in V^*$, defines a hyperplane $\mathbb{P}(\operatorname{Ann}\xi) \subset \mathbb{P}_n$. As an equation on ξ for a fixed v, it defines a hyperplane in $\mathbb{P}_n^\times$ formed by all hyperplanes in $\mathbb{P}_n$ passing through v. For every $k = 0, 1, \dots, n$ there is the canonical involutive[1] bijection $L \leftrightarrow \operatorname{Ann} L$ between projective subspaces of dimension k in $\mathbb{P}_n$ and projective subspaces of dimension $(n-k-1)$ in $\mathbb{P}_n^\times$. It is called *the projective duality*. For a given $L = \mathbb{P}(U) \subset \mathbb{P}_n$, the dual subspace $\operatorname{Ann} L \overset{\text{def}}{=} \mathbb{P}(\operatorname{Ann} U) \subset \mathbb{P}_n^\times$ consists of all hyperplanes in $\mathbb{P}_n$ containing L. The projective duality reverses inclusions: $L \subset H \iff \operatorname{Ann} L \supset \operatorname{Ann} H$, and sends intersections to linear joins, and vise versa. This allows to translate the theorems true for $\mathbb{P}_n$ to the dual statements about the dual figures in $\mathbb{P}_n^\times$. The latter may look quite dissimilar to the original. For example, the collinearity of 3 points in $\mathbb{P}_n$ is translated as the existence of codimension-2 subspace common for 3 hyperplanes in $\mathbb{P}_n^\times$.

2.3.1 The polar mapping. For a smooth quadric $Q = V(q)$, the correlation $\widehat{q} : V \to V^*$ is an isomorphism. The induced linear projective isomorphism $\overline{q} : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ is called the *polar mapping* or *the polarity* provided by quadric Q. The polarity sends a point $p \in \mathbb{P}_n$ to the hyperplane

$$\Pi_p = \operatorname{Ann} \overline{q}(p) = \{ x \in \mathbb{P}(V) \mid \widetilde{q}(p, x) = 0 \},$$

which cuts apparent contour of Q viewed from p in accordance with Corollary 2.2. The hyperplane Π_p and point p are called *the polar* and *pole* of one other with respect to Q. If $p \in Q$, then $\Pi_p = T_p Q$ is the tangent plane to Q at p. Note that a lies on the polar of b if and only if b lies on the polar of a, because the condition $\widetilde{q}(a, b) = 0$ is symmetric. Such points a, b are called *conjugated* with respect to the quadric $Q = V(q)$.

Proposition 2.3

Let a line (ab) intersect a smooth quadric Q in two distinct points c, d different from a, b. Then a, b are conjugated with respect to Q if and only if they are harmonic to c, d.

Proof. Chose some homogeneous coordinate $x = (x_0 : x_1)$ on the line $\ell = (ab) = (cd)$. The intersection $Q \cap \ell = \{c, d\}$ considered as a quadric in ℓ is the zero set of quadratic form

$$q(x) = \det(x, c) \cdot \det(x, d),$$

whose polarization is $\widetilde{q}(x, y) = \frac{1}{2}\big(\det(x, c) \cdot \det(y, d) + \det(y, c) \cdot \det(x, d)\big)$. Thus, $\widetilde{q}(a, b) = 0$ means that $\det(a, c) \cdot \det(b, d) = -\det(b, c) \cdot \det(a, d)$, i.e., $[a, b, c, d] = -1$. □

Proposition 2.4

Let $G, Q \subset \mathbb{P}_n$ be two quadrics with Gram matrices A, Γ in some basis of $\mathbb{P}_n$. If G is smooth, then the polar mapping of G sends Q to the quadric $Q_G^\times \subset \mathbb{P}_n^\times$ which has the Gram matrix $A_\Gamma^\times = \Gamma^{-1} A \Gamma^{-1}$ in the dual basis of $\mathbb{P}_n^\times$. Note that $\operatorname{rk} Q_G^\times = \operatorname{rk} Q$.

Proof. Write the homogeneous coordinates in $\mathbb{P}_n$ as row vectors x and dual coordinates in $\mathbb{P}_n^\times$ as column vectors ξ. The polarity $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n^\times$ provided by G sends $x \in \mathbb{P}_n$ to $\xi = \Gamma x^t$. Since Γ is invertible, x is recovered from ξ as $x = \xi^t \Gamma^{-1}$. When x runs through the quadric $x A x^t = 0$, the corresponding ξ fills the quadric $\xi^t \Gamma^{-1} A \Gamma^{-1} \xi = 0$. □

[1] That is, inverse to itself: $\operatorname{Ann}\operatorname{Ann} L = L$.

Corollary 2.3
The tangent spaces to a smooth quadric $Q \subset \mathbb{P}_n$ form the smooth quadric $Q^\times \subset \mathbb{P}_n^\times$. The Gram matrices of Q, $Q^\times$ in dual bases of $\mathbb{P}_n$, $\mathbb{P}_n^\times$ are inverse to each other.

Proof. Put $G = Q$ and $\Gamma = A$ in Proposition 2.4. □

2.3.2 Polarities over non-closed fields. If $\Bbbk$ is not algebraically closed, then there are non-singular quadratic forms $q \in S^2V^2$ with $V(q) = \varnothing$. However, their polarities $\overline{q} : \mathbb{P}(V) \to \mathbb{P}(V^*)$, that is, the bijective correspondences between points and hyperplanes, are non-trivial anyway.

Exercise 2.6. Describe geometrically the polarity with respect to «imaginary circle» $x^2+y^2 = -1$ in the Euclidean plane $\mathbb{R}^2$.

Thus, the polarities are much more informative than the quadrics. The quadric is recovered from its polarity as the set of all points lying on the own polars, i.e., the self-conjugated points. It follows from Theorem 1.1 that two polarities coincide if and only if the corresponding quadratic forms are proportional. Thus, the polarities on $\mathbb{P}_n = \mathbb{P}(V)$ stay in bijection with the points of projective space $\mathbb{P}(S^2V^*) = \mathbb{P}_{\frac{n(n+3)}{2}}$. Somewhat erroneous, the latter is called *the space of quadrics* in $\mathbb{P}(V)$. The quadrics $Q \subset \mathbb{P}_n$ passing through a given point $p \in \mathbb{P}_n$ form a hyperplane in the space of quadrics, because the equation $q(p) = 0$ is linear homogeneous in $q \in \mathbb{P}(S^2V^*)$.

Proposition 2.5
Every collection of $n(n + 3)/2$ points in $\mathbb{P}_n$ lies on some quadric.

Proof. Any $n(n + 3)/2$ hyperplanes in $\mathbb{P}_{\frac{n(n+3)}{2}}$ have a non empty intersection. □

Proposition 2.6
Over an infinite field, two nonempty smooth quadrics coincide if and only if their equations are proportional.

Proof. If $V(q_1) = V(q_2)$ in $\mathbb{P}(V)$, then two polarities $\overline{q}_1, \overline{q}_2 : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ coincide in all points of the quadrics. It follows from Corollary 1.1 on p. 12 and Exercise 2.7 below that the correlation maps $\widehat{q}_1, \widehat{q}_2 : V \xrightarrow{\sim} V^*$ and therefore the Gram matrices are proportional. □

Exercise 2.7. Check that over an infinite field, every nonempty smooth quadric $\mathbb{P}_n$ contains $n + 2$ points such that no $n + 1$ of them lie within a hyperplane.

2.4 Conics. Plane quadrics are called *conics*. For $\mathbb{P}_2 = \mathbb{P}(V)$, the space of conics $\mathbb{P}(S^2V^*) = \mathbb{P}_5$. Conics of rank 1 are called *a double lines*. In appropriate coordinates, such a conic has the equation $x_0^2 = 0$. It is totally singular, i.e., has no smooth points at all. By Theorem 2.1 on p. 18, a conic S of rank 2 is the linear join of the singular point $s \in S$ and a smooth quadric $S \cap \ell$ within a line $\ell \not\ni s$. By Example 2.1 on p. 17, $S \cap \ell$ either consists of two distinct points or is empty. In the first case, S is the union of two lines intersecting at the singular point s. Such a conic is called *split*. If $S \cap \ell = \varnothing$, then $S = \{s\}$ consists of the singular point only. For example, the conic $x_0^2 + x_1^2 = 0$ in $\mathbb{P}(\mathbb{R}^3)$ is of this sort. Over an algebraically closed field, there are no such conics, certainly.

Lemma 2.2 (rational parametrization of non-empty smooth conic)
Every non-empty smooth conic $C \subset \mathbb{P}_2$ over any field $\Bbbk$ with char $\Bbbk \neq 2$ admits a rational quadratic parametrization, i.e., there exist homogeneous quadratic polynomials $\varphi_0, \varphi_1, \varphi_2 \in \Bbbk[t_0, t_1]$ such that the map $\varphi : \mathbb{P}_1 \to \mathbb{P}_2, (t_0 : t_1) \mapsto \big(\varphi_0(t_0, t_1) : \varphi_1(t_0, t_1) : \varphi_2(t_0, t_1)\big)$, establishes a bijection between $\mathbb{P}_1$ and C.

PROOF. Given a point $p \in C$, a required parametrization is provided by the projection $\varphi : \ell \xrightarrow{\sim} C$ of an arbitrary line $\ell \not\ni p$ from p onto C. For every $t \in \ell$, the line (pt) intersects C at p and one more point, which coincides with p, if $(pt) = T_pC$, and differs from p for all other t. In the first case we put $\varphi(t) = a$. For all other t, the second intersection point can be written as $t + \lambda p$, where $\lambda \in \Bbbk$, and satisfies the equation $\widetilde{q}(t + \lambda p, t + \lambda p) = 0$, which is equivalent to $q(t) = -2\lambda\widetilde{q}(t, p)$. Thus, the map $\varphi : \ell \xrightarrow{\sim} C$ takes $t \in \ell$ to $\varphi(t) = q(t) \cdot p - 2q(p, t) \cdot t \in C$. □

EXERCISE 2.8. Verify that the right hand side of the latter formula equals p for $t = T_pC \cap \ell$, and make sure that φ is described in coordinates by a triple of quadratic homogeneous polynomials in the coordinates of t as required.

LEMMA 2.3
The intersection $C \cap D$ of a smooth conic C with a curve D of degree d in $\mathbb{P}_2$ either consists of at most $2d$ points or coincides with C.

PROOF. Let $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$, $(t_0 : t_1) \mapsto \big(\varphi_0(t_0, t_1) : \varphi_1(t_0, t_1) : \varphi_2(t_0, t_1)\big)$ be a rational quadratic parameterization of C, and $D = V(f)$ for some homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. The values of parameter t corresponding to the intersection point $C \cap D$ satisfy the equation $f\big(\varphi_0(t), \varphi_1(t), \varphi_2(t)\big) = 0$, whose left hand side is either the zero polynomial or a non-zero homogeneous polynomial of degree $2d$. In the first case $C \subset D$. In the second case the equation has at most $2d$ solutions in $\mathbb{P}_1$. □

PROPOSITION 2.7
Any 5 points in $\mathbb{P}_2$ lie on a conic. Such a conic C is unique if and only if every 4 of the points are non-collinear. If every 3 of the points are non-collinear, the conic C is smooth.

PROOF. The first statement is exactly Proposition 2.5 for $n = 2$. Let a line ℓ pass through some 3 of the given points. Then any conic C passing through the given points contains ℓ. If the remaining two pints a, b do not lie on ℓ, then $C = \ell \cup (ab)$ is unique. If $a \in \ell$, then for any line $\ell' \ni b$, the split conic $\ell \cup \ell'$ contains all five given points. If any 3 of the given points are non-collinear, then every conic passing through the 5 given points is smooth, because a singular conic is either a line, or a pair of lines, or a point. Since two different smooth conics have at most 4 intersection points by Lemma 2.3, a smooth conic passing through 5 points is unique. □

COROLLARY 2.4
Any 5 lines without triple intersections in $\mathbb{P}_2$ do touch a unique smooth conic.

PROOF. This is projectively dual to the last statement in Proposition 2.7. □

2.5 Quadratic surfaces. The space of quadrics in $\mathbb{P}_3 = \mathbb{P}(V)$ is $\mathbb{P}(S^2V^*) = \mathbb{P}_9$. In particular, any 9 points in $\mathbb{P}_3$ lie on some quadric.

EXERCISE 2.9. Show that any 3 lines in $\mathbb{P}_3$ lie on a quadric.

A quadratic surface of rank 1 is called *a double plane*. It is totally singular and has the equation $x_0^2 = 0$ in appropriate coordinates on $\mathbb{P}_3$. A quadratic surface S of rang 2 either is *a split quadric*, i.e., a union of two planes intersecting along the singular line $\ell = \operatorname{Sing} S$, or is exhausted by the singular line, and the latter case is impossible over an algebraically closed field.

EXERCISE 2.10. Prove this.

A quadratic surface $S \subset \mathbb{P}_3$ of rank 3 is called *a simple cone*. It is ruled by the lines (sp), where $s \in S$ is the singular point and t runs through a smooth conic $C = S \cap \Pi$ laying in a plane $\Pi \not\ni s$. Note that C may be empty as soon the ground field is not algebraically closed. In this case $S = \{s\}$ is exhausted by the singular point. If $C \neq \varnothing$, the linear span of C is the whole Π.

EXERCISE 2.11. Convince yourself that the lines laying on a simple cone with vertex s over a smooth conic C are exhausted by the lines (st), $t \in C$.

As a byproduct of the previous discussion, we get

PROPOSITION 2.8
Every 3 mutually non-intersecting lines in $\mathbb{P}_3$ lie on a smooth quadratic surface. □

Over an algebraically closed field, all smooth quadrics in $\mathbb{P}_3$ are congruent modulo the linear projective automorphisms of $\mathbb{P}_3$. The most convenient model of the smooth quadric is described below.

2.5.1 The Segre quadric. Let U be a vector space of dimension 2. Write $W = \mathrm{End}(U)$ for the space of linear maps $F : U \to U$, and consider $\mathbb{P}_3 = \mathbb{P}(W)$. A choice of basis in U identifies W with the space $\mathrm{Mat}_2(\Bbbk)$ of 2×2 matrices. The quadric

$$S \overset{\text{def}}{=} \{F : \in \mathrm{End}(U) \mid \det F = 0\} = \left\{ \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \;\middle|\; x_0x_3 - x_1x_2 = 0 \right\} \subset \mathbb{P}_3 \tag{2-3}$$

is called *the Segre quadric*. It is formed by endomorphisms of rank 1 considered up to proportionality. The image of an operator $F : U \to U$ of rank 1 has dimension 1 and is spanned by a non zero vector $v \in U$, uniquely determined by F up to proportionality. The value of F on an arbitrary vector $u \in U$ equals $F(u) = \xi(u) \cdot v$, where $\xi \in U^*$ is a linear form such that $\mathrm{Ann}\,\xi = \ker F$. Note that ξ is uniquely determined by F and $v \in \mathrm{im}\, F \smallsetminus 0$. Conversely, for any non-zero $v \in U$, $\xi \in U^*$ the operator

$$\xi \otimes v : U \to U\,, \quad u \mapsto \xi(u)\,v$$

has rank 1, its image is spanned by v, and the kernel equals $\mathrm{Ann}\,\xi$. Thus, we have the well defined injective map

$$s : \mathbb{P}(U^*) \times \mathbb{P}(U) \hookrightarrow \mathbb{P}\,\mathrm{End}(U)\,, \quad (\xi, v) \mapsto \xi \otimes v\,, \tag{2-4}$$

whose image coincides with the Segre quadric (2-3). This map is called *the Segre embedding*.

The rows of any 2×2 matrix of rank 1 are proportional, as well as the columns. The matrices with a fixed ratio ([row 1] : [row 2]) $= (t_0 : t_1)$ or ([column 1] : [column 2]) $= (\xi_0 : \xi_1)$ form a vector subspace of dimension 2 in $W = \mathrm{Mat}_2(\Bbbk)$. After the projectivization these subspaces turns to the two families of lines ruling the Segre quadric. These lines are the images of «coordinate lines» $\mathbb{P}_1^\times \times v$ and $\xi \times \mathbb{P}_1$ on the product $\mathbb{P}_1^\times \times \mathbb{P}_1 = \mathbb{P}(U^*) \times \mathbb{P}(U)$ under the bijection $\mathbb{P}_1^\times \times \mathbb{P}_1 \xrightarrow{\sim} S$ provided by the Segre embedding (2-4). Indeed, the operator $\xi \otimes v$ build from from $\xi = (\xi_0 : \xi_1) \in U^*$ and $v = (t_0 : t_1) \in U$ has the matrix

$$\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \cdot \begin{pmatrix} \xi_0 & \xi_1 \end{pmatrix} = \begin{pmatrix} \xi_0 t_0 & \xi_1 t_0 \\ \xi_0 t_1 & \xi_1 t_1 \end{pmatrix} \tag{2-5}$$

with the prescribed ratios $(t_0 : t_1)$ and $(\xi_0 : \xi_1)$ between the rows and columns respectively. Since the Segre map $\mathbb{P}_1^\times \times \mathbb{P}_1 \xrightarrow{\sim} S$ is bijective, the incidence relations among coordinate lines in $\mathbb{P}_1^\times \times \mathbb{P}_1$ are the same as among their images in S. That is, within each ruling family, all the lines are mutually non-intersecting, every two lines from different ruling families are intersecting, and each point on the Segre quadric is an intersection point of exactly two lines from different families.

EXERCISE 2.12. Prove that all lines $\ell \subset S$ are exhausted by these two ruling families.

PROPOSITION 2.9 (CONTINUATION OF PROPOSITION 2.8)
A smooth quadric Q passing through a triple ℓ_1, ℓ_2, ℓ_3 of mutually non-intersecting lines in $\mathbb{P}_3$, as in Proposition 2.8, is ruled by all those lines in $\mathbb{P}_3$ that do intersect all the lines ℓ_i. In particular, this quadric is unique.

PROOF. If a line ℓ intersects all the lines ℓ_i, it has at least 3 distinct points on Q and therefore lies on Q. On the other side, for any point $a \in Q$ not laying on the lines ℓ_i, the tangent plane T_aQ intersects every line ℓ_i at some point $p_i \neq a$. Since the line (ap_i) touches Q at a, it lies on Q. Thus, all three lines (ap_i) lie on the conic $Q \cap T_aQ$. Hence, at least two of them, say (ap_1), (ap_2), coincide. If p_3 does not belong to the line $\ell = (ap_1) = (ap_2)$, then the tangent plane $T_{p_3}Q$ intersects ℓ at a point b different from a and all p_i's. The line $(p_3b) \subset Q$ by the same reason as above. Thus, Q contains the triangle abp_3 formed by 3 distinct lines ℓ, (ap_3), and (ab). Hence, Q contains the whole plane spanned by this triangle[1].

EXERCISE 2.13. Show that a smooth quadric in $\mathbb{P}_3$ can not contain a plane.

Therefore, the points a, p_1, p_2, p_3 are collinear, that is, a lies on a line intersecting all the lines ℓ_i. □

EXERCISE 2.14. Given 4 mutually non-intersecting lines in $\mathbb{P}_3$, how many lines intersect them all?

2.6 Linear subspaces lying on a smooth quadric. A smooth quadric Q is called *k-planar*, if there is a projective subspace $L \subset Q$ of dimension $\dim L = k$ and Q does not contain a subspace of higher dimension. By the definition, the planarity of the empty quadric is -1. Thus, the quadrics of planarity 0 are non-empty and do not contain lines.

PROPOSITION 2.10
The planarity of a smooth quadric $Q \subset \mathbb{P}_n$ does not exceed $\dim Q/2 = (n-1)/2$.

PROOF. Let $\mathbb{P}_n = \mathbb{P}(V)$ and $L = \mathbb{P}(W) \subset Q = V(q)$ for some non-singular quadratic form $q \in S^2V^*$ and a vector subspace $W \subset V$. Since $q|_W = 0$, the correlation $\hat{q} : V \xrightarrow{\sim} V^*$ sends W into $\mathrm{Ann}(W)$. Since $\hat{q}$ is injective, $\dim(W) = \dim \hat{q}(W) \leqslant \dim \mathrm{Ann}\, W = \dim V - \dim W$. Thus, $2\dim W \leqslant \dim V$ and $2\dim L \leqslant n-1$. □

LEMMA 2.4
For any smooth quadric Q and hyperplane Π, the intersection $\Pi \cap Q$ either is a smooth quadric in Π or has exactly one singular point $p \in \Pi \cap Q$. The latter happens if and only if $\Pi = T_pQ$.

PROOF. Let $Q = V(q) \subset \mathbb{P}(V)$, $\Pi = \mathbb{P}(W)$. Since $\dim\ker\left(\hat{q}|_W\right) = \dim\left(W \cap \hat{q}^{-1}(\mathrm{Ann}\, W)\right) \leqslant \dim \hat{q}^{-1}(\mathrm{Ann}\, W) = \dim \mathrm{Ann}\, W = \dim V - \dim W = 1$, the quadric $\Pi \cap Q \subset \Pi$ has at most one singular point. If $\mathrm{Sing}\, Q = \{p\} \neq \varnothing$, then the kernel $\ker \hat{q}|_W \subset W$ has dimension 1 and is spanned by p. Thus, $\mathrm{Ann}(\hat{q}(p)) = W$, that is, $T_pQ = \Pi$. Vice versa, if $\Pi = T_pQ = \mathbb{P}(\mathrm{Ann}\,\hat{q}(p))$, then $p \in \mathrm{Ann}\,\hat{q}(p)$ belongs to the kernel of the restriction of $\hat{q}$ on $\mathrm{Ann}\,\hat{q}$. □

[1]Because for every point of the plane except for the vertexes of triangle, every line passing through this point intersects all three lines ℓ, (ap_3), and (ab).

Proposition 2.11
Let $Q \subset \mathbb{P}_{n+1}$ be a smooth quadric of dimension n. For every $1 \leqslant m \leqslant n/2$, the projective subspaces of dimension m laying in Q and passing through a given point $p \in Q$ stay in bijection with all projective subspaces of dimension $m-1$ laying on a smooth quadric of dimension $n-2$ cut out of Q by any hyperplane $H \subset T_pQ$ complementary to p within the tangent hyperplane $T_pQ \simeq \mathbb{P}_{n-1}$.

Proof. Every projective subspace $L \subset Q$ of dimension m passing through $p \in Q$ lies inside the intersection $Q \cap T_pQ$, which is the singular quadric in $\mathbb{P}_{n-1} = T_pQ$ with just one singular point p by Lemma 2.4. It accordance with Theorem 2.1 on p. 18, the quadric $Q \cap T_pQ \subset \mathbb{P}_{n-1}$ is the cone ruled by lines (pa), where a runs through the smooth quadric Q' cut out of Q by a hyperplane $H \subset \mathbb{P}_{n-1}$ not passing through p. Thus, the subspaces $L \subset Q \cap T_pQ$ of dimension n are exactly the linear joins of p with the subspaces $L' = L \cap H = L \cap Q'$ of dimension $m-1$ laying on Q'. □

Corollary 2.5
For any two distinct points a, b on a smooth quadric Q and all $0 \leqslant m \leqslant \dim Q/2$ there is a bijection between the subspaces of dimension m laying on Q and passing through the points a and b respectively. In particular, a projective subspace of dimension k laying on a smooth k-planar quadric can be drown through every point of the quadric.

Proof. If $b \notin T_aQ$, then $H = T_aQ \cap T_bQ$ does not pass through a, b and lies in the both tangent spaces T_aQ, T_bQ as a hyperplane. By Proposition 2.11, the sets of projective subspaces $L \subset Q$ of dimension m passing through a and b respectively both stay in bijection with the subspaces $L' \subset Q \cap H$ of dimension $m-1$. If $b \in T_aQ$, pick up a point $c \in Q \smallsetminus (T_aQ \cup T_bQ)$ and repeat the previous arguments twice for the pairs a, c and c, b. □

Corollary 2.6
A smooth quadric of dimension n over an algebraically closed field is $[n/2]$-planar.

Proof. This holds for $n = 0, 1, 2$. Then we use Proposition 2.11 and induction in n. □

§3 Working examples: lines and conics on the plane

3.1 Homographies. A linear projective isomorphism between two projective lines is called *a homography*. An important example of homography is provided by *a perspective* $o : \ell_1 \xrightarrow{\sim} \ell_2$, the central projection of a line $\ell_1 \subset \mathbb{P}_2$ to another line $\ell_2 \subset \mathbb{P}_2$ from a point $o \notin \ell_1 \cup \ell_2$, see fig. 3◇1.

EXERCISE 3.1. Make sure that a perspective is a homography.

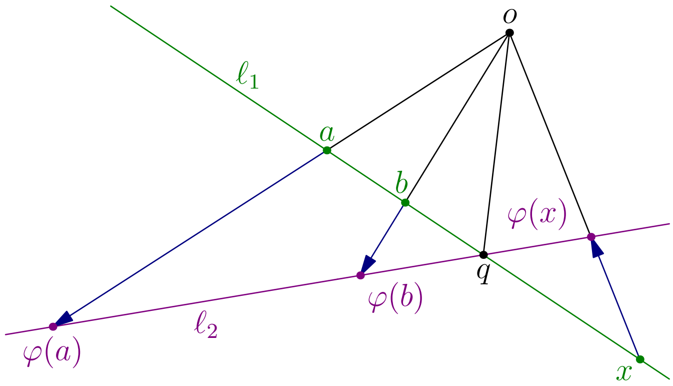


Fig. 3◇1. The perspective $o : \ell_1 \xrightarrow{\sim} \ell_2$.

A homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ is a perspective if and only if it sends the intersection point $\ell_1 \cap \ell_2$ to itself. Indeed, choose two distinct points $a, b \in \ell_1 \smallsetminus \ell_2$ and put $o = (a\varphi(a)) \cap (b\varphi(b))$ as on fig. 3◇1. Then the perspective $o : \ell_1 \xrightarrow{\sim} \ell_2$ sends the points a, b, $\ell_1 \cap \ell_2$ to $\varphi(a)$, $\varphi(b)$, $\ell_1 \cap \ell_2$. Thus, it coincides with φ if and only if φ maps the intersection of lines to itself.

3.1.1 The cross-axis. Given two lines $\ell_1, \ell_2 \subset \mathbb{P}_2$ intersecting at the point $q = \ell_1 \cap \ell_2$, then for any line $\ell \subset \mathbb{P}_2$ and points $b_1 \in \ell_1$, $b_2 \in \ell_2$ the composition of perspectives

$$(b_1 : \ell \to \ell_2) \circ (b_2 : \ell_1 \to \ell) \tag{3-1}$$

takes $b_1 \mapsto b_2$, $\ell_1 \cap \ell \mapsto q$, $q \mapsto \ell_2 \cap \ell$, see fig. 3◇2.

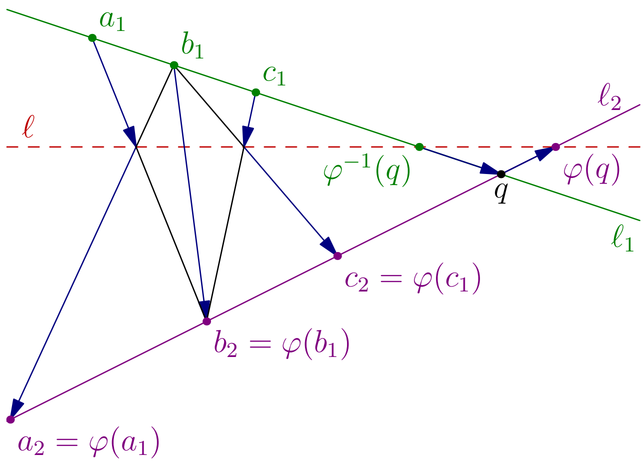


Fig. 3◇2. The cross-axis of a homography.

Every homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ admits a decomposition (3-1) in which the point $b_1 \in \ell_1$ can be chosen arbitrarily, $b_2 = \varphi(b_1)$, and the line ℓ is uniquely predicted by φ and does not depend on the choice of $b_1 \in \ell_1$. Indeed, fix some distinct points $a_1, b_1, c_1 \in \ell_1 \smallsetminus \ell_2$ and write $a_2, b_2, c_2 \in \ell_2$ for their images under φ. Put ℓ as the line joining the cross-intersections $(a_1 b_2) \cap (b_1 a_2)$ and $(c_1 b_2) \cap (b_1 c_2)$. Then the composition (3-1) sends a_1, b_1, c_1 to a_2, b_2, c_2 and therefore coincides with φ, see fig. 3⋄2. If we repeat this argument for the ordered triple c_1, a_1, b_1 instead of a_1, b_1, c_1, then we get the decomposition $\varphi = (a_1 : \ell' \to \ell_2) \circ (a_2 : \ell' \to \ell)$, where ℓ' joins the cross-intersections $(a_1 c_2) \cap (c_1 a_2)$ and $(b_1 a_2) \cap (a_1, b_2)$, see fig. 3⋄3. Since both lines ℓ, ℓ' pass through the points[1] $(b_1 a_2) \cap (a_1, b_2)$, $\varphi(q)$, $\varphi^{-1}(q)$, we conclude that $\ell = \ell'$. Hence, all the cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$, where $x \neq y$ are running through ℓ_1, lie on the same line ℓ, which is uniquely determined by this property.

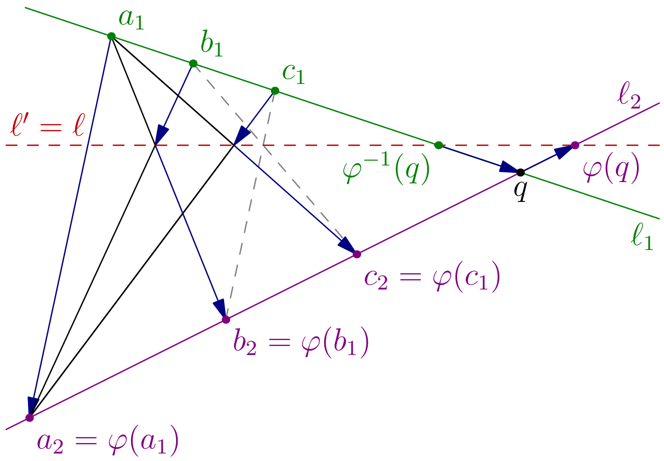


Fig. 3⋄3. Coincidence $\ell' = \ell$.

Definition 3.1 (the cross-axis of homography)
Given a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$, the line ℓ drown by cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$ as $x \neq y$ run through ℓ_1 is called the *cross-axis* of φ.

Remark 3.1. The cross-axis of non-perspective homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ is well defined as the line joining $\varphi(\ell_1 \cap \ell_2)$ and $\varphi^{-1}(\ell_1 \cap \ell_2)$, which are distinct. If φ is a perspective, then the point $\varphi(\ell_1 \cap \ell_2) = \varphi^{-1}(\ell_1 \cap \ell_2) = \ell_1 \cap \ell_2$ still lies on the cross-axis but does not fix it uniquely.

Exercise 3.2. Let a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ send 3 given points $a_1, b_1, c_1 \in \ell_1$ to 3 given points $a_2, b_2, c_2 \in \ell_2$. Using only the ruler, construct $\varphi(x)$ for a given $x \in \ell_1$.

Lemma 3.1
Let $\Bbbk$ be an algebraically closed field of zero characteristic. If a bijection

$$\varphi : \mathbb{P}_1(\Bbbk) \smallsetminus \{\text{finite set of points}\} \xrightarrow{\sim} \mathbb{P}_1(\Bbbk) \smallsetminus \{\text{finite set of points}\}$$

can be described in some affine chart with a local coordinate t by a formula

$$\varphi : t \mapsto \varphi_0(t)/\varphi_1(t), \quad \text{where } \varphi_0, \varphi_1 \in \Bbbk[t], \tag{3-2}$$

then φ is the restriction of a unique homography $\mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$.

[1] Note that the latter two coincide as soon φ is a perspective.

PROOF. In the homogeneous coordinates $(x_0 : x_1)$ such that $t = x_0 / x_1$, the formula (3-2) can be rewritten[1] as $\varphi : (x_0 : x_1) \mapsto \big(f_0(x_0, x_1) : f_1(x_0, x_1)\big)$, where $f_0, f_1 \in \Bbbk[x_0, x_1]$ are non-proportional homogeneous polynomials of the same degree d. Write $\mathbb{P}_d$ for the projectivization of space of homogeneous polynomials of degree d in x_0, x_1. As soon as a point $\vartheta = (\vartheta_0 : \vartheta_1) \in \mathbb{P}_1$ has a unique preimage under φ, the polynomial $h_\vartheta(x_0, x_1) = \vartheta_1 f(x_0, x_1) - \vartheta_0 g(x_0, x_1)$ has just one root in $\mathbb{P}_1$. Since $\Bbbk$ is algebraically closed, h_ϑ is the proper d th power of a linear form, that is, lies on the Veronese curve[2] $C_d \subset \mathbb{P}_d$. On the other hand, the polynomial h_ϑ runs through the line $(f_0, f_1) \subset \mathbb{P}_d$ as ϑ runs through $\mathbb{P}_1$. Since $\mathbb{P}_1(\Bbbk)$ is infinite, we conclude that the Veronese curve has infinitely many intersections with the line (f_0, f_1). But for $d \geqslant 2$, any 3 distinct points of C_d are non-collinear[3]. Hence, $d = 1$ and $\varphi \in \mathrm{PGL}_2(\Bbbk)$. □

3.1.2 Homographies provided by conics. Let a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ send an ordered triple of distinct points $a_1, b_1, c_1 \in \ell_1 \smallsetminus \ell_2$ to $a_2, b_2, c_2 \in \ell_2$. If the lines $(a_1 a_2)$, $(b_1 b_2)$, $(c_1 c_2)$ meet all together at some point p, then φ coincides with the perspective $p : \ell_1 \xrightarrow{\sim} \ell_2$, and this happens if and only if $\varphi(q) = q$, see fig. 3⋄4.

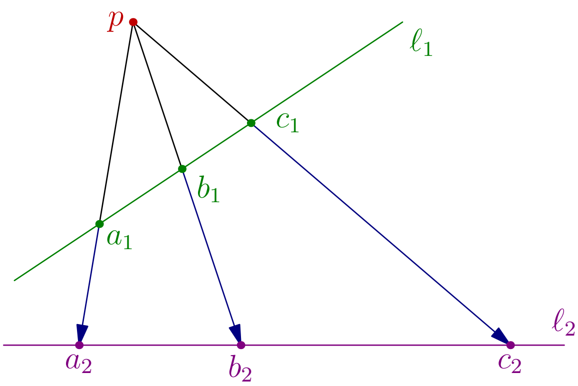


Fig. 3⋄4. Perspective $p : \ell_1 \to \ell_2$.

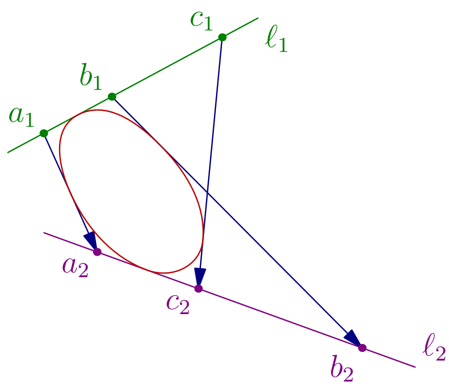


Fig. 3⋄5. Homography $C : \ell_1 \to \ell_2$.

If the lines $(a_1 a_2)$, $(b_1 b_2)$, $(c_1 c_2)$ are not concurrent, then any 3 of the 5 lines ℓ_1, ℓ_2, (a_1, a_2), (b_1, b_2), (c_1, c_2) are not concurrent, and there exists a unique smooth conic C touching all these 5 lines by Corollary 2.4 on p. 21, see fig. 3⋄5. In this case, the homography φ is provided by the tangent lines to C, i.e., $y = \varphi(x)$ if and only if the line (xy) is tangent to C. Indeed, the map $C : \ell_1 \to \ell_2$, which sends $x \in \ell_1$ to the intersection point of ℓ_2 with the tangent line from x to C other than ℓ_1, is obviously bijective.

EXERCISE 3.3. Convince yourself that this map satisfies Lemma 3.1.

We conclude that $C : \ell_1 \to \ell_2$ is a homography that acts on a_1, b_1, c_1 exactly as φ.

Thus, every homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ is either a perspective $p : \ell_1 \xrightarrow{\sim} \ell_2$ provided by some point $p \notin \ell_1 \cup \ell_2$ or a homography $C : \ell_1 \to \ell_2$ provided by a smooth conic C touching the both lines ℓ_1, ℓ_2. In both cases, the point p and conic C are uniquely predicted by φ. The perspective $p : \ell_1 \xrightarrow{\sim} \ell_2$ can be treated as a degeneration of the non-perspective homography $C : \ell_1 \xrightarrow{\sim} \ell_2$ arising when C splits in two lines crossing at the centre of perspective. However these two lines can

[1] Perhaps, after a modification of the finite set on which φ is undefined.
[2] See n° 1.3.3 on p. 9.
[3] See n° 1.3.3 on p. 9.

be chosen in many ways: any two lines joining the corresponding points are fitted in the picture. Note also that the image and preimage of $\ell_1 \cap \ell_2$ under the homography $C : \ell_1 \xrightarrow{\sim} \ell_2$ are the points of contact $\ell_2 \cap C$ and $\ell_1 \cap C$ respectively.

Proposition 3.1 (inscribed-circumscribed triangles)
Two triangles $\triangle\, a_1 b_1 c_1$ and $\triangle\, a_2 b_2 c_2$ are both inscribed in some smooth conic Q' if and only if they are both circumscribed about some smooth conic Q''.

Proof. Let 6 points $a_1, b_1, c_1, a_2, b_2, c_2$ lie on a smooth conic Q' like in fig. 3⋄6. Put $\ell_1 = (a_1 b_1)$, $\ell_2 = (a_2 b_2)$ and write $c_2 : \ell_1 \xrightarrow{\sim} Q'$ for the projection of ℓ_1 onto Q' from c_1 and $c_1 : Q' \xrightarrow{\sim} \ell_2$ for the projection of Q' onto ℓ_2 from c_2. The composition $[c_1 : Q' \xrightarrow{\sim} \ell_2] \circ [c_2 : \ell_1 \xrightarrow{\sim} Q'] : \ell_1 \xrightarrow{\sim} \ell_2$ is a non-perspective homography sending $a_1 \mapsto p$, $q \mapsto b_2$, $r \mapsto a_2$, $b_1 \mapsto s$. Let Q'' be a smooth conic whose tangent lines join the homographic points. Then Q'' is obviously inscribed in the both triangles. The opposite implication is projectively dual to just proven. □

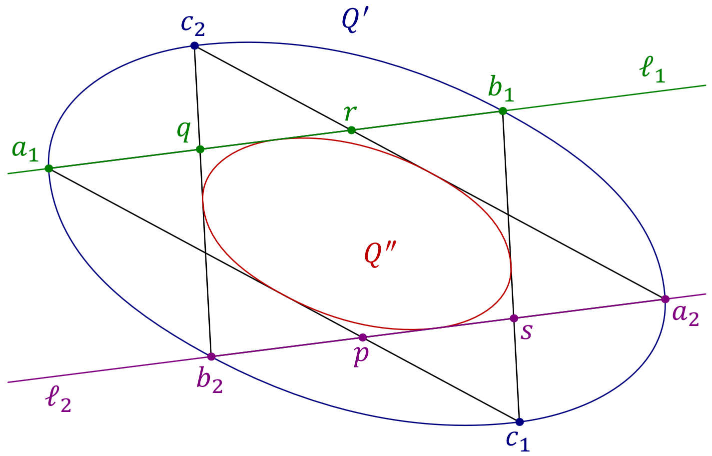


Fig. 3⋄6. Inscribed circumscribed triangles.

Corollary 3.1 (Poncelet's porism for triangles)
Assume that a triangle $\triangle\, a_1 b_1 c_1$ is simultaneously inscribed in a smooth conic Q' and circumscribed about a smooth conic Q''. Then every point of Q' except for a finite set is a vertex of triangle simultaneously inscribed in Q' and circumscribed about Q''.

Proof (see fig. 3⋄6). For any $a_2, b_2, c_2 \in Q'$ such that $(a_2 b_2)$, $(a_2 c_2)$ are two different tangent lines to Q'', the triangles $\triangle\, a_1 b_1 c_1$ and $\triangle\, a_2 b_2 c_2$ are both circumscribed about some smooth conic C by Proposition 3.1. Since C touches 5 lines $(a_1 b_1)$, $(b_1 c_1)$, $(c_1 a_1)$, $(a_2 b_2)$, $(a_2 c_2)$, it coincides with Q'' by Corollary 2.4 on p. 21. □

3.1.3 Homographic pencils of lines. Projectively dual version of the construction from n° 3.1.2 deals with a homography $\varphi : p_1^\times \xrightarrow{\sim} p_2^\times$ between two pencils of lines in $\mathbb{P}_2$ passing through the points p_1 and p_2 respectively. Let φ sent 3 distinct lines $\ell_1', \ell_2', \ell_3' \ni p_1$ other than $(p_1 p_2)$ to the lines $\ell_1'', \ell_2'', \ell_3'' \ni p_1$. Write $q_i = \ell_i' \cap \ell_i''$, $i = 1, 2, 3$, for the intersection points of corresponding lines. Since every 4 points from p_1, p_2, q_1, q_2, q_3 are non-collinear, there exists the unique conic

C_φ passing through these 5 points, see fig. 3◇7 and fig. 3◇8 below. Provided by this conic is the homography $C : p_1^\times \simeq p_2^\times$ sending $(p_1 p) \mapsto (p_2 p)$ for all $p \in C_\varphi$.

Exercise 3.4. Use Lemma 3.1 on p. 26 to convince yourself that this map is actually a homography.

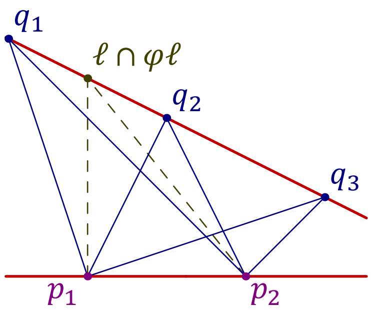


Fig. 3◇7. Perspective homography $\varphi : p_1^\times \to p_2^\times$.

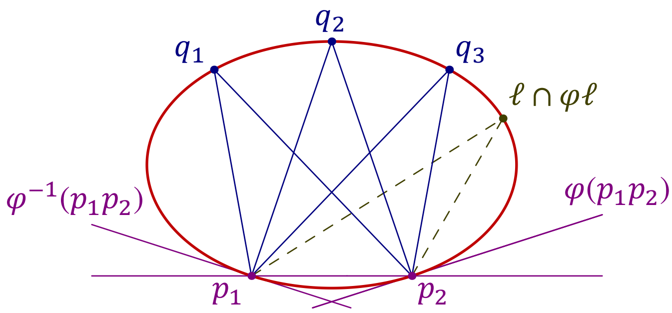


Fig. 3◇8. Non-perspective homography $\varphi : p_1^\times \to p_2^\times$.

Since this homography takes $\ell_i' \mapsto \ell_i''$ for $i = 1, 2, 3$, it coincides with φ, see. fig. 3◇8. The homography provided by a smooth conic C_φ takes $T_{p_1}C_\varphi \mapsto (p_1p_2)$ and $(p_1p_2) \mapsto T_{p_2}C_\varphi$. The conic C_φ splits if and only if the points q_1, q_2, q_3 are collinear or, equivalently, when the line (p_1p_2) goes to itself. In this case $C_\varphi = (p_1p_2) \cup (q_iq_j)$ and the homography is a perspective, see fig. 3◇7. In a contrast with n° 3.1.2, the split conic C_φ is uniquely determined by the perspective φ in this case.

Example 3.1 (tracing conic by the ruler)
Let C be a conic drawn through 5 given points $p_1, p_2, \dots, p_5$ no 3 of which are collinear. The points of C can be constructed by the ruler as follows. Draw the lines $\ell_1 = (p_2p_5)$, $\ell_2 = (p_2p_4)$ and mark the point $p = (p_1p_4) \cap (p_3p_5)$, see fig. 3◇9.

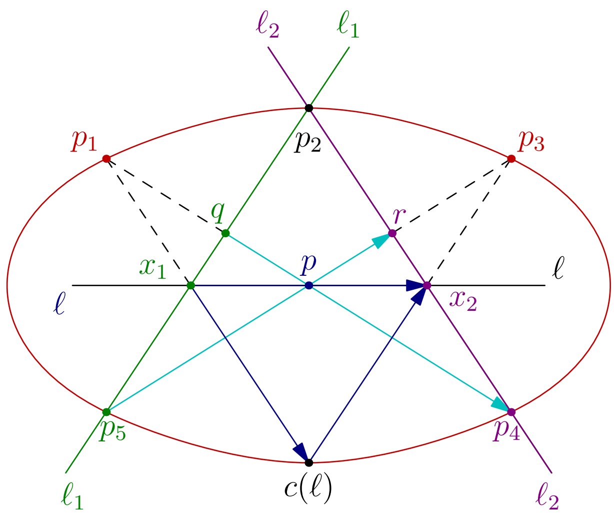


Fig. 3◇9. Tracing a conic by a ruler.

The perspective $p : \ell_1 \simeq \ell_2$ is decomposed as the projection $p_1 : \ell_1 \simeq C$ of ℓ_1 onto C from p_1 followed by projection $p_3 : C \simeq \ell_2$ from C onto ℓ_2 from p_3.

Exercise 3.5. Check this by comparing the action on points $p_2, p_5, q \in \ell_1$, see fig. 3◇9.

Thus, for any line $\ell \ni p$, the lines joining p_1, p_2 with the intersection points $x_1 = \ell \cap \ell_1$, $x_2 = \ell \cap \ell_2$ are crossing at the point $c(\ell) = (p_1 x_1) \cap (p_2 x_2) \in C$, see fig. 3⋄9. As ℓ turns about p, the point $c(\ell)$ draws the conic C.

THEOREM 3.1 (PASCAL'S THEOREM)
Six points $p_1, p_2, \dots, p_6$ no 3 of which are collinear lie on a smooth conic if and only if 3 intersection points[1] $x = (p_3 p_4) \cap (p_6 p_1)$, $y = (p_1 p_2) \cap (p_4 p_5)$, $z = (p_2 p_3) \cap (p_5 p_6)$ are collinear.

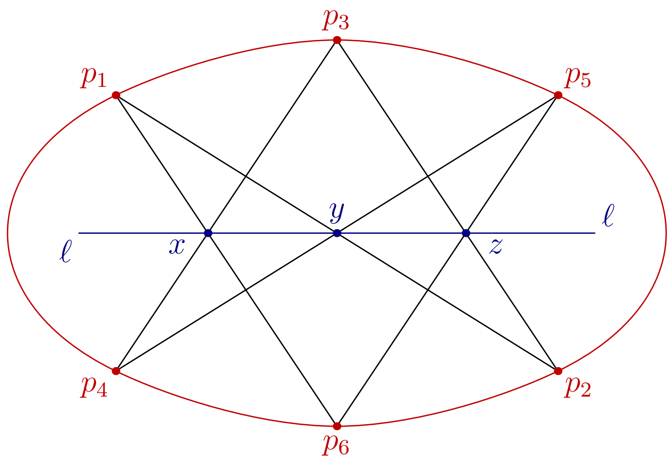


Fig. 3⋄10. The hexogram of Pascal.

PROOF. Let $\ell_1 = (p_3 p_4)$, $\ell_2 = (p_3 p_2)$, see fig. 3⋄10. Assume that $z \in (xy)$. Then the perspective $y : \ell_1 \to \ell_2$ takes $x \mapsto z$ and is decomposed[2] as $\big(p_5 : C \simeq \ell_2\big) \circ (p_1 : \ell_1 \simeq C)$, where C is the smooth conic passing trough $p_1, p_2, \dots, p_5$. Thus, $p_6 = (p_5 z) \cap (p_3 x) \in C$. Conversely, if $(p_5 z) \cap (p_3 x) \in C$, then the above composition takes $x \mapsto z$. Hence, the perspective $y : \ell_1 \to \ell_2$ also sends $x \mapsto z$ forcing $z \in (xy)$. □

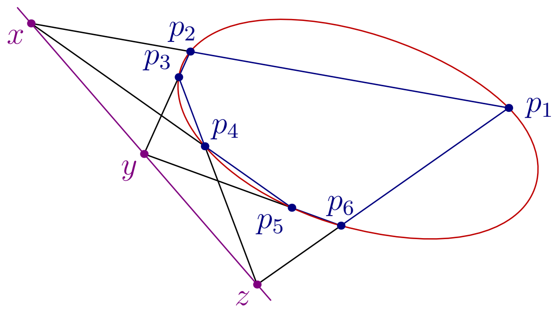


Fig. 3⋄11. Inscribed hexagon.

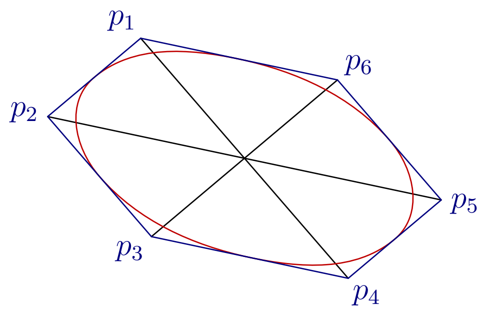


Fig. 3⋄12. Circumscribed hexagon.

COROLLARY 3.2 (BRIANCHON'S THEOREM)
A hexagon $p_1, p_2, \dots, p_6$ is circumscribed about a non-singular conic if and only if «the main diagonals» $(p_1 p_4)$, $(p_2 p_5)$, $(p_3 p_6)$ are concurrent, see fig. 3⋄12.

PROOF. This is dual to Theorem 3.1, comp. fig. 3⋄11 and fig. 3⋄12. □

[1] They can be thought of as intersection points of «the opposite sides» of hexagon $p_1, p_2, \dots, p_6$.
[2] See Example 3.1 on p. 29.

3.2 Internal geometry of a smooth conic. In this section we assume on default that the ground field $\Bbbk$ is algebraically closed and $\mathrm{char}(\Bbbk) \neq 2$. Dual projective lines $\mathbb{P}_1 = \mathbb{P}(U)$, $\mathbb{P}_1^\times = \mathbb{P}(U^*)$ are naturally identified by the canonical homography provided by projective duality:

$$\delta : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1^\times , \quad v \mapsto \mathrm{Ann}\, v . \tag{3-3}$$

In coordinates, it takes a point $(p_0 : p_1) \in \mathbb{P}_1$ to the linear form $\det(p,t) = p_0 t_1 - p_1 t_0$, whose coordinates in the dual basis of $\mathbb{P}_1^\times$ are $(-p_1 : p_0)$. The plane $\mathbb{P}_2 = \mathbb{P}(S^2 U^*)$ can be thought[1] of as the space of non-ordered pairs of possibly coinciding points in $\mathbb{P}_1 = \mathbb{P}(U)$ by mapping a pair of points $p = (p_0 : p_1)$, $q = (q_0 : q_1)$ on $\mathbb{P}_1$ to the binary quadratic form with roots $\{p, q\}$:

$$\begin{aligned} f_{pq}(t_0, t_1) &= \det \begin{pmatrix} p_0 & t_0 \\ p_1 & t_1 \end{pmatrix} \det \begin{pmatrix} q_0 & t_0 \\ q_1 & t_1 \end{pmatrix} = \\ &= p_0 q_0 \cdot t_0^2 - (p_0 q_1 + p_1 q_0) \cdot t_0 t_1 + p_1 q_1 \cdot t_1^2 \in S^2 U^* . \end{aligned} \tag{3-4}$$

We will often misuse the notations and write $\{p, q\} \in \mathbb{P}_2$ for the quadratic form (3-4). Pairs $\{p, t\} \in \mathbb{P}_2$, where $p \in \mathbb{P}_1$ is fixed and t runs through $\mathbb{P}_1$, form a line in $\mathbb{P}_2$. This line consists of all $f \in S^2(U^*)$ such that $f(p) = 0$. Pairs of coinciding points $\{p, p\} \in \mathbb{P}_2$ form the smooth Veronese conic $C \subset \mathbb{P}_2$. The above line $\{p, t\}$ is tangent to C at the point $\{p, p\}$, certainly. Thus, the pair of tangent lines to C drown through a point $\{p, q\} \notin C$ is formed by $\{p, t\}$, $\{q, t\}$, where $t \in \mathbb{P}_1$, which meet C at the points $\{p, p\}$, $\{q, q\}$.

The Veronese conic stays in the natural bijection with $\mathbb{P}_1$ provided by the Veronese map[2]

$$\mathbb{P}_1 \hookrightarrow \mathbb{P}_2 , \quad p \mapsto \{p, p\} .$$

In coordinates, it takes a point $(p_0 : p_1) \in \mathbb{P}_1$ to the binary quadratic form $x_0 t_0^2 + 2 x_1 t_0 t_1 + x_2 t_2^2$ with coefficients

$$(x_0 : x_1 : x_2) = (p_0^2 : -p_0 p_1 : p_1^2) . \tag{3-5}$$

We refer the ratio $(p_0 : p_1)$ as *the internal homogeneous coordinate* of the point $\{p, p\}$ on the Veronese conic, and define the cross-ratio of four points $\{p_i, p_i\}$, $i = 1, \dots, 4$, on C as $[p_1, p_2, p_3, p_4]$ on $\mathbb{P}_1$. Note that the internal homogeneous coordinates on C are predicted by a choice of basis in $\mathbb{P}_1$ whereas the cross-ratio does not depend on a choice of coordinates.

As soon as $\Bbbk$ is algebraically closed and $\mathrm{char}\,\Bbbk \neq 2$, every smooth conic D on the plane can be identified with the Veronese conic C by means of linear projective automorphism of the plane. This allows to introduce internal homogeneous coordinates and the cross-ratio on D. We would like to verify that different choices of the linear projective automorphism $\varphi : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ such that $\varphi(D) = C$ do not change the cross-ratio and lead to invertible linear changes of the internal homogeneous coordinates. To this aim, let us redefine the cross-ratio more geometrically.

Definition 3.2 (the cross-ratio on a smooth conic)
Given an ordered quadruple of different points a_1, a_2, a_3, a_4 on a smooth conic D, consider a point $c \in D$ other than given. The cross-ratio of lines $\big[(ca_1), (ca_2), (ca_3), (ca_4)\big]$ in the pencil $c^\times$ of lines passing through c is called *the cross-ratio* of points a_i on D.

---

[1] See n° 1.3.3 on p. 9.

[2] Note that this map differs from the map $\mathbb{P}_1^\times \hookrightarrow \mathbb{P}_2$, described in formula (1-5) on p. 10 and Example 1.4, by composing with the latter with duality isomorphism $\mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1^\times$ from (3-3).

EXERCISE 3.6. Prove that the cross-ratio does not depend on the choice of c and is preserved by linear projective automorphisms of the plane.

Since the parameterization (3-5) of the Veronese conic $C : x_0x_2 = x_1^2$ can be obtained by composing the projection[1] $a : \ell \xrightarrow{\sim} C$ of the line $\ell : x_2 = 0$ onto C from the point $a = (0 : 0 : 1) \in C$

EXERCISE 3.7. Verify that this projection takes $(p_0 : p_1 : 0) \mapsto (p_0^2 : p_0p_1 : p_1^2)$.

with the homography $\ell \xrightarrow{\sim} \ell$, $(p_0 : p_1 : 0) \mapsto (p_0 : -p_1 : 0)$, Definition 3.2 agrees with the previous definition of homogeneous coordinates and cross-ratio on the Veronese conic.

PROPOSITION 3.2
The smooth conic D passing through 5 points $p_1, p_2, \dots, p_5$ no 3 of which are collinear consists of all the points $p \in \mathbb{P}_2$ such that $\big[(pp_1), (pp_2), (pp_3), (pp_4)\big] = \big[(p_5p_1), (p_5p_2), (p_5p_3), (p_5p_4)\big]$.

PROOF. It follows from Exercise 3.7 that the equality between cross-ratios holds for all points $p \in D$. Consider any point $p \in \mathbb{P}_2$ for which the equality holds, and write Q for the conic passing through p, p_1, p_2, p_3, p_5. Provided by Q is the homography[2] $Q : p^\times \to p_5^\times$ sending a line (pq) to the line (p_5q) for all $q \in Q$. It takes $(pp_i) \mapsto (p_5p_i)$ for $i = 1, 2, 3$. Since $[(pp_1), (pp_2), (pp_3), (pp_4)] = [(p_5p_1), (p_5p_2), (p_5p_3), (p_5p_4)]$, the line (pp_4) goes to the line (p_5p_4). Hence, $p_4 \in Q$ and therefore $Q = D$, because D is the only conic passing through $p_1, p_2, \dots, p_5$. Thus, $p \in D$. □

EXERCISE 3.8. Given 5 points $p, q, a, b, c \in \mathbb{P}_2$ any 3 of which are non-collinear, consider the homography of pencils $\gamma : p^\times \to q^\times$ sending the lines (pa), (pb), (pc) to the lines (qa), (qb), (qc). Describe the locus of intersection points $\ell \cap \gamma(\ell)$ for $\ell \in p^\times$.

3.2.1 Homographies on a smooth conic. A bijection $\varphi : C \xrightarrow{\sim} C$ provided by an invertible linear change of internal homogeneous coordinates on a smooth conic C is called *a homography*. It follows from Lemma 3.1 on p. 26 that every rational bijection of the form

$$\varphi : C \smallsetminus \{\text{finite set of points}\} \xrightarrow{\sim} C \smallsetminus \{\text{finite set of points}\} \tag{3-6}$$

$$(t_0 : t_1) \mapsto \big(f_0(t_0/t_1) : f_1(t_0/t_1)\big), \tag{3-7}$$

where $f_0, f_1 \in \Bbbk[t_0, t_1]$, is the restriction of unique homography $C \xrightarrow{\sim} C$. For any two ordered triples of distinct points on C there exists a unique homography sending one triple to the other. A bijection $C \xrightarrow{\sim} C$ is a homography if and only if it preserves the cross-ratio on C.

PROPOSITION 3.3
Every homography $\gamma : C \xrightarrow{\sim} C$ on a smooth conic $C \subset \mathbb{P}_2$ admits the unique extension to a linear projective automorphism $\widetilde{\gamma} : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ of the plane. Conversely, any linear projective automorphism $\varphi : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ such that $\varphi(C) = C$ induces the homography $\varphi|_C : C \xrightarrow{\sim} C$.

PROOF. Chose 5 distinct points $p_1, p_2, \dots, p_5 \in C$, let $\gamma : C \xrightarrow{\sim} C$ be a homography, and put $q_i = \gamma(p_i)$. There exists a unique linear projective automorphism $\widetilde{\gamma} : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ such that $\widetilde{\gamma}(p_i) = q_i$ for $1 \leqslant i \leqslant 4$. Since $\widetilde{\gamma}$ preserves the cross-ratio in the corresponding pencils of lines, the cross-ratio of lines (q_5, q_i), $1 \leqslant i \leqslant 4$, in the pencil $q_5^\times$ equals the cross-ratio of lines (p_5, p_i), $1 \leqslant i \leqslant 4$, in the pencil $p_5^\times$. Since the latter equals the cross-ratio of lines (p_5, q_i), $1 \leqslant i \leqslant 4$, in the same pencil, because $\gamma : C \xrightarrow{\sim} C$ is the homography and preserves the cross-ratio on C. Thus, for any 5

[1] See Example 1.5 on p. 11.
[2] See n° 3.1.3 on p. 28.

points $p_1, p_2, \dots, p_5 \in C$ the cross-ratios of lines passing through p_1, p_2, p_3, p_4 in the pencils $p_5^\times$ and $\widetilde{\gamma}(p_5)^\times$ coincide. Hence, $\widetilde{\gamma}(p_5) \in C$ by Proposition 3.2. The converse statement follows from Exercise 3.6. □

EXAMPLE 3.2 (INVOLUTIONS)
A self-inverse homography $\sigma : C \to C$, $\sigma^2 = \mathrm{Id}_C$, is called *an involution* of the conic C. The identity involution $\sigma = \mathrm{Id}_C$ is referred to as *trivial*.

Let an involution $\sigma : C \to C$ interchange a' with a'' and b' with b'' for some mutually different points $a', a'', b', b'' \in C$, as on fig. 3⋄13. Consider the intersection point $s = (a'a'') \cap (b'b'')$. Provided by s is the involution $\sigma_s : C \xrightarrow{\sim} C$ swapping the pair of intersection points $\ell \cap C$ on every line $\ell \ni s$.

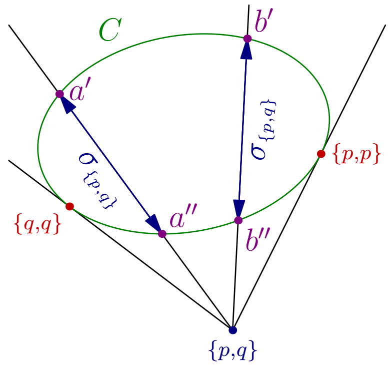


Fig. 3⋄13. Involution of conic.

EXERCISE 3.9. Convince yourself that the map σ_s satisfies the conditions of Lemma 3.1 on p. 26, and therefore it is a homography.

Since the actions of σ_s and σ on 4 points a', a'', b', b'' coincide, $\sigma = \sigma_s$. In particular, every non-trivial involution has exactly two distinct fixed points[1], the points of contact of two tangent lines to C coming from s. If C is identified with the Veronese conic, the fixed points of involution $\sigma_{p,q}$ are $\{p,p\}$ and $\{q,q\}$. We conclude that every involutive homography $\gamma : \mathbb{P}_1 \to \mathbb{P}_1$ over algebraically closed field has exactly two distinct fixed points $p, q \in \mathbb{P}_1$, and $\gamma(a) = b$ if and only if the points $\{a,a\}$, $\{b,b\}$, $\{p,q\}$ are collinear in $\mathbb{P}_2$.

EXERCISE 3.10. Verify that the latter is equivalent to the harmonicity $[p,q,a,b] = -1$.

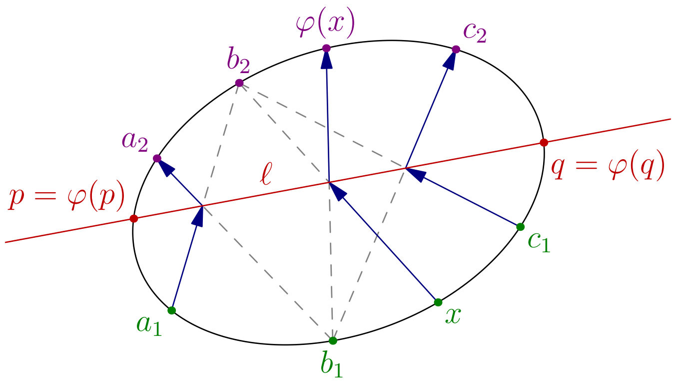


Fig. 3⋄14. The cross-axis of a homography on conic.

3.2.2 The cross-axis of a homography on conic. A homography $\varphi : C \xrightarrow{\sim} C$ sending a_1, b_1, c_1 to $a_2, b_2, c_2 \in C$ is decomposed as projection $b_2 : C \to \ell$ followed by projection $b_1 : \ell \to C$, where ℓ is the line joining cross-intersections $(a_1b_2) \cap (b_1a_2)$ and $(c_1b_2) \cap (b_1c_2)$, see fig. 3⋄14. Since the intersection points $\ell \cap C$ are exactly the fixed points[2] of φ, the line ℓ is uniquely predicted by φ

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[1] Recall, we assume that $\Bbbk$ is algebraically closed and char $\Bbbk \neq 2$.

[2] In particular, this forces φ to have either two distinct fixed points or just one fixed «double point», and the latter means that ℓ is tangent to C at the fixed point. Note that in both cases ℓ is uniquely recovered from the set of fixed points.

and does not depend on the choice of points $a_1, b_1, c_1 \in C$. In other words, the intersection point of crossing lines $(x, \varphi(y)) \cap (y, \varphi(x))$ draws the line ℓ as $x \neq y$ run through C. This gives another proof for the Pascal theorem[1]: the opposite sides of hexagon $a_1 c_2 b_1 a_2 c_1 b_2$ inscribed in C are the crossing lines for the homography sending a_1, b_1, c_1 to a_2, b_2, c_2, and therefore their intersection points lie on the cross-axix ℓ of this homography.

The cross axis of a homography $\varphi : C \to C$ can be easily drawn by the ruler as soon the action of φ on some triple of points is known. This allows to construct the image $\varphi(z)$ of any given point $z \in C$, and to find the fixed points of φ using only the ruler. In particular, given a smooth conic C and point s in $\mathbb{P}_2$, it is not hard to draw the tangent lines to C from s by means of the ruler only: one could either construct the fixed points of involution $\sigma_s : C \to C$ provided by the pencil $s^{\times}$, as on fig. 3⋄15, or use more elegant method based on Exercise 3.11 below.

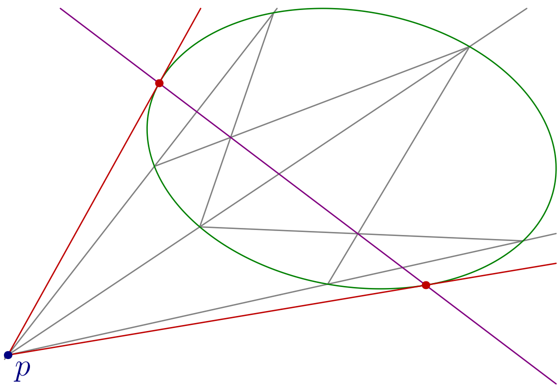


Fig. 3⋄15. Drawing the tangent lines.

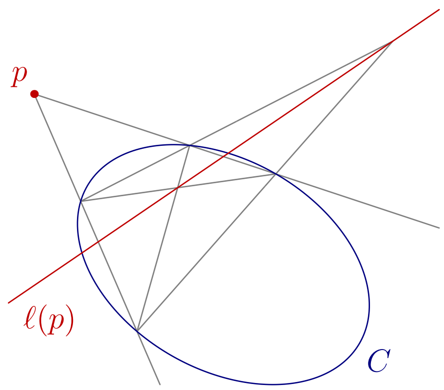


Fig. 3⋄16. Drawing the polar.

Exercise 3.11 (Steiner's construction). Shown on fig. 3⋄16 is the construction of polar line $\ell(p)$ for a point p with respect to a conic C due to Jacob Steiner[2] (1796 – 1863) and using only the ruler. Explain how and why does it work.

3.3 Pencils of conics. Recall[3] that lines in the space of conics $\mathbb{P}(S^2V^*)$ on the plane $\mathbb{P}_2 = \mathbb{P}(V)$ are called *pencils of conics*. A pencil $L \subset \mathbb{P}(S^2V^*)$ is uniquely described by any pair of distinct conics $C_0 = V(f_0)$, $C_1 = V(f_1)$ from L and consists of the conics $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1)$, where $\lambda = (\lambda_0 : \lambda_1) \in \mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$. The intersection $B = C_0 \cap C_1$ is called *the base set* of the pencil. It does not depend on the choice of basis $C_0, C_1 \in L$, because every conic $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1) \in L$ contains $B = V(f_0) \cap V(f_1)$ for any two distinct conics $C_0 = V(f_0)$, $C_1 = V(f_1)$ in L.

The polynomial $\chi_{(f_0 f_1)}(t_0, t_1) \stackrel{\text{def}}{=} \det(t_0 f_0 + t_1 f_1) \in \Bbbk[t_0, t_1]$ is called *the characteristic polynomial* of the pencil with respect to the base conics C_0, C_1. This is a cubic homogeneous polynomial. Up to multiplication by non zero constants, it does not dependent on a choice of basis in V used for the evaluation of determinant. However, in a contrast with the base set, the characteristic polynomial depends on a choice of basis in the pencil, and a change of basis leads to an invertible linear change of variables (t_0, t_1). Thus, an invariant of the pencil is not the characteristic polynomial

[1] See Theorem 3.1 on p. 30.

[2] See *J. Steiner*. «Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises: als Lehrgegenstand auf höheren Unterrichts-Anstalten und zur praktischen Benutzung», Ostwald's Klassiker der exakten Wissenschaften, vol. 60.

[3] See n° 1.3.2 on p. 9.

itself but the combinatorial structure of its zero set in $\mathbb{P}_1$. Over algebraically closed field, the latter is either the whole $\mathbb{P}_1$, or one point of multiplicity 3, or a pair of distinct points of multiplicities 1 and 2, or a triple of distinct points, each of multiplicity 1. In the first case, the pencil is called *degenerated*; in the latter case, it is called *simple*. Thus, a pencil is degenerated if and only it consists of singular conics. A non-degenerated pencil over algebraically closed field can contain 1, 2, or 3 degenerated conics, and $\operatorname{Sing} C_0 \cap \operatorname{Sing} C_1 = \varnothing$ for any two different conics C_0, C_1 in the pencil, because a vector $v \in \ker \widehat{f}_0 \cap \ker \widehat{f}_1$ belongs to $\ker(\lambda_0 \widehat{f}_\lambda + \lambda_1 \widehat{f}_1)$ for all $\lambda \in \mathbb{P}_1$. The base set of a non-degenerated pencil over algebraically closed field can consist of 1, 2, 3, or 4 points.

Lemma 3.2
For every conic $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1)$ in a non-degenerated pencil, $\dim \operatorname{Sing} C_\lambda$ is strictly less than the maximal power of $\det(\lambda, t) = \lambda_0 t_1 - \lambda_1 t_0$ dividing the characteristic polynomial $\chi_{(f_0 f_1)}(t_0, t_1)$ in $\Bbbk[t_0, t_1]$.

Proof. Let D be an arbitrary conic of the pencil, and C a smooth conic. Fix a basis in V such that the Gram matrix of C is the identity matrix E, and write A for the Gram matrix of D. Then the conics in pencil (CD) become the Gram matrices $tE + A$, where $t \in \Bbbk$ is a coordinate on affine line $(CD) \smallsetminus C$. The conic D appears for $t = 0$. We have to show that $\dim \ker A$ can not exceed the maximal power of t dividing $\det(tE + A) = t^3 + t^2\delta_1(A) + t\delta_2(A) + \delta_3(A)$, where $\delta_k(A)$ is the sum of principal $k \times k$ minors in A. This is obvious, because all minors of order $> 3 - k$ in A vanish as soon $\operatorname{rk} A \leqslant 3 - k$. □

Exercise 3.12. Prove that a non-degenerated pencil of conics contains at most one double line.

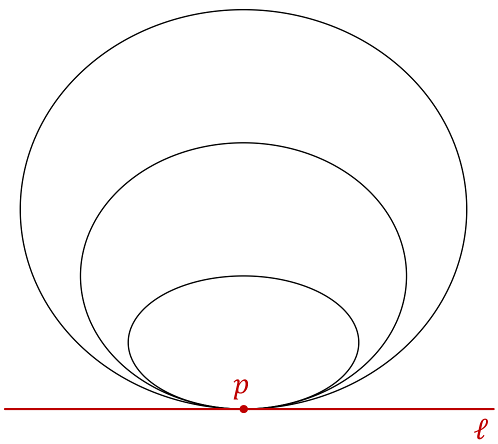


Fig. 3⋄17. A pencil with 1 base point.

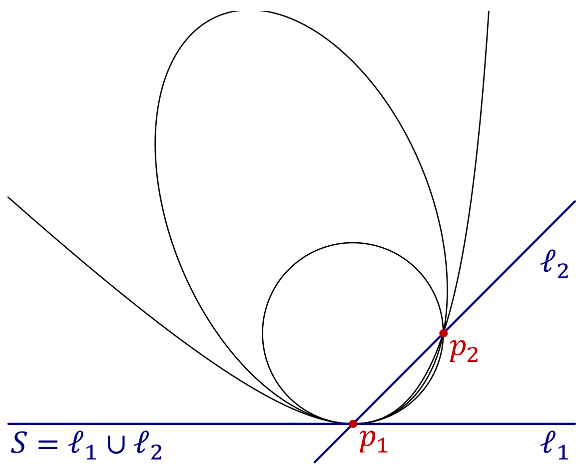


Fig. 3⋄18. A pencil with 2 base points and 1 singular conic.

Example 3.3 (non-degenerated pencil with just one base point)
If the base set of a non-degenerated pencil consists of just one point p, then the only singular conic in the pencil is the double line tangent to any smooth conic of the pencil at the point p. Thus, such a pencil is spanned by a smooth conic $C \ni p$ and the double line $\ell = T_p C$. Note that any two smooth conics in such a pencil have the unique intersection point and share the common tangent line at this point, see fig. 3⋄17.

Example 3.4 (non-degenerated pencils with two base points)
If the base set of a pencil consists of two points $p_1 \neq p_2$, then a singular conic in such pencil has to be either the double line $\ell = (p_1p_2)$ or a split conic $\ell_1 \cup \ell_2$ such that $p_1 \in \ell_1$, $p_2 \in \ell_2$ and either p_1, p_2 both differ from $\ell_1 \cap \ell_2$, as on fig. 3⋄19, or $p_1 = \ell_1 \cap \ell_2$, $p_2 \neq \ell_1 \cap \ell_2$, as on fig. 3⋄18.

In the latter case the split conic $\ell_1 \cap \ell_2$ is the only singular conic in the pencil. All the other conics are smooth, touch the line ℓ_1 at p_1, and pass through p_2 like on fig. 3⋄18. In particular, any two smooth conics in such a pencil have exactly two different intersection points p_1, p_2 and share the same tangent line at p_1.

The first two possibilities for a singular conic, i.e., the double line $\ell = (p_1p_2)$ or a split conic $\ell_1 \cup \ell_2$ such that $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2 \in \ell_2 \smallsetminus \ell_2$, can be realized in a pencil with 2 base points only simultaneously.

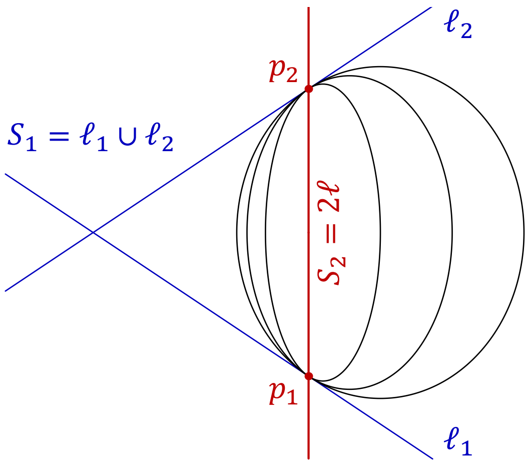


Fig. 3⋄19. A pencil with 2 base points and 2 singular conics S_1, S_2.

Exercise 3.13. Prove that all conics in $\mathbb{P}_2$ that touch two given lines ℓ_1, ℓ_2 at two given points $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2 \in \ell_2 \smallsetminus \ell_1$ form a pencil with exactly two singular conics: the double line $\ell = (p_1p_2)$ and the split conic $\ell_1 \cup \ell_2$.

Both lines ℓ_1, ℓ_2 are uniquely recovered from the double line ℓ and any smooth conic C of the pencil as the tangent lines to C at the intersection points $C \cap \ell$.

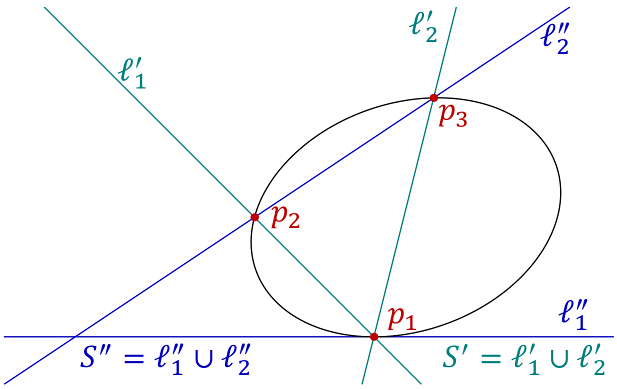


Fig. 3⋄20. A pencil with 3 base poins has 2 singular conics.

Example 3.5 (non-degenerated pencil with three base points)
If the base set of a pencil consists of 3 distinct points p_1, p_2, p_3, then these points are not collinear[1]. Hence, such a pencil does not contain a double line. For any split conic $\ell_1 \cup \ell_2$ in the pencil, there are two possibilities: either $p_1 = \ell_1 \cap \ell_2$, $p_2 \in \ell_1 \smallsetminus \ell_2$, $p_3 \in \ell_2 \smallsetminus \ell_1$ or $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2, p_3 \in \ell_2 \smallsetminus \ell_1$. On fig. 3⋄20, the first happens for the lines ℓ'_1, ℓ'_2, the second for the lines ℓ''_1, ℓ''_2. If the pencil contains $\ell''_1 \cup \ell''_2$, then every smooth conic from the pencil touches ℓ''_1 at p_1. Note that the split

[1]Otherwise the line passing through them would intersect every smooth conic of the pencil in 3 distinct points.

conic $\ell_1' \cup \ell_2'$ satisfies this property.

EXERCISE 3.14. Prove that all conics passing through 3 given distinct points a, b, c and touching a given line $\ell \ni c$ form a pencil containing exactly 2 singular conics: $(ab) \cup \ell$ and $(ac) \cup (bc)$.

If the pencil contains $\ell_1' \cup \ell_2'$, then all smooth conics in the pencil also have to share the same tangent line at the point p_1, because a line $\ell \ni p_1$ tangent to a smooth conic $C \ni p_1$ touches at p_1 every conic D from the pencil spanned by C and $\ell_1' \cup \ell_2'$. Thus, such a pencil is described by Exercise 3.14 as well.

EXAMPLE 3.6 (SIMPLE PENCIL OF CONICS)
A pencil of conics over algebraically closed field is simple if and only if it contains three distinct singular conics. Each of these singular conics splits by Lemma 3.2, and does not pass trough the singular points of two others. Therefore every pair of singular conics has 4 intersection points any 3 of which are non-collinear, see fig. 3◇21. These 4 points form the base set of pencil.

EXERCISE 3.15. Prove that all conics passing through 4 given points a, b, c, d no 3 of which are collinear form a simple pencil containing exactly 3 singular conics formed by the pairs of opposite sides in quadrangle $abcd$.

Thus, a simple pencil of conics is uniquely determined by its base points a, b, c, d. In homogeneous coordinates $x = (x_0 : x_1 : x_2)$ on $\mathbb{P}_2$, the equations of conics from this pencil can be written as

$$\frac{\det(x, a, b) \cdot \det(x, c, d)}{\det(x, a, d) \cdot \det(x, b, c)} = \frac{\lambda_0}{\lambda_1},$$

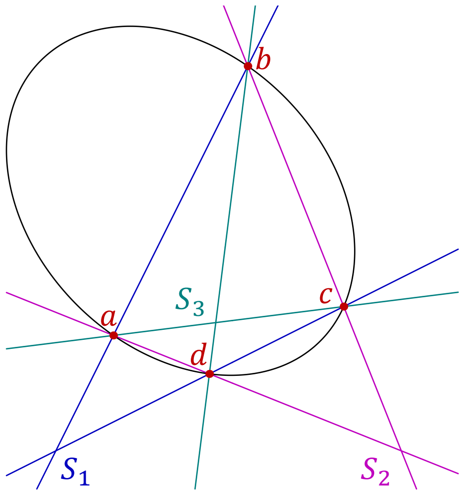


Fig. 3◇21. 3 singular conics and 4 base points of a simple pencil.

where $\lambda = (\lambda_0 : \lambda_1)$ runs through $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$.

All the previous examples of pencils can be viewed as degenerations of a simple pencil appearing when some of the base points stick together. For $a, b \to p_1$, $c = p_2$, $d = p_3$, we get the pencil on fig. 3◇20. For $a, b \to p_1$, $c, d \to p_2$, we come to the pencil on на fig. 3◇19. When $a, b, c \to p_1$, $d = p_2$, we get fig. 3◇18. Finally, on fig. 3◇17, all 4 base points are collapsed to one point p.

3.3.1 The hypersurface of singular conics. The singular conics in $\mathbb{P}_2 = \mathbb{P}(V)$ form a cubic hypersurface $S = V(\det)$ in the space $\mathbb{P}_5 = \mathbb{P}(S^2)$ of all conics. The roots of characteristic polynomial $\chi_{(f_0 f_1)}(t_0, t_1)$ correspond to the intersection points of S with the line $L = (C_0 C_1)$ spanned by conics $C_0 = V(f_0)$, $C_1 = V(f_1)$. The character of intersection $S \cap L$ completely determines the geometric properties of the pencil L. A simple pencil L intersects S in 3 distinct points with the multiplicity 1 at each point. If L touches S at a smooth point of S and intersects S with the multiplicity 1 in one more point, then the pencil L looks as on fig. 3◇20, where the split conic with singularity at a base point of L corresponds to the touch point of L with S. If L passes through a singular point of S and intersects S once more in another point, then L looks as on fig. 3◇19, where the double line corresponds to the singular intersection point of L and S. If L intersects S with the multiplicity 3 in one smooth point of S, the pencil looks as on fig. 3◇18. The most degenerated pencil shown on fig. 3◇17 is provided by a line L intersecting S with the multiplicity 3 in one singular point of S.

4.1 Tensor products and Segre varieties. Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over a field $\Bbbk$. A map $\varphi : V_1 \times V_2 \times \dots \times V_n \to W$ is called *multilinear*, if it is linear in each argument when all the other are fixed:

$$\varphi(\,\dots\,,\,\lambda v' + \mu v''\,,\,\dots) = \lambda\,\varphi(\,\dots\,,\,v'\,,\,\dots) + \mu\,\varphi(\,\dots\,,\,v''\,,\,\dots)\,.$$

Multilinear maps $V_1 \times V_2 \times \dots \times V_n \to W$ form a vector space denoted $\mathrm{Hom}(V_1, V_2, \dots, V_n; W)$. As soon some bases $e_1, e_2, \dots, e_m \in W$ and $e^{(i)}_1, e^{(i)}_2, \dots, e^{(i)}_{d_i} \in V_i$, $1 \leqslant i \leqslant n$, are fixed, every multilinear map $\varphi \in \mathrm{Hom}(V_1, V_2, \dots, V_n; W)$ can be uniquely described by the values on all collections of basis vectors:

$$\varphi\big(\,e^{(1)}_{\alpha_1}\,,\,e^{(2)}_{\alpha_2}\,,\,\dots\,,\,e^{(n)}_{\alpha_n}\big) = \sum_\nu a^{(\alpha_1,\alpha_2,\dots,\alpha_n)}_\nu \cdot e_\nu \,\in W\,,$$

that is, by $m \cdot \prod d_\nu$ constants $a^{(\alpha_1,\alpha_2,\dots,\alpha_n)}_\nu \in \Bbbk$, which can be organized in the matrix of dimension $(n+1)$ and size[1] $m \times d_1 \times d_2 \times \dots \times d_n$. The multilinear map φ corresponding to such a matrix sends a collection of vectors $v_1, v_2, \dots, v_n$, where $v_i = \sum_{\alpha_i=1}^{d_i} x^{(i)}_{\alpha_i} e^{(i)}_{\alpha_i} \in V_i$ for $1 \leqslant i \leqslant n$, to the vector

$$\varphi(v_1, v_2, \dots, v_n) = \sum_{\nu=1}^{m} \Big(\sum_{\alpha_1,\alpha_2,\dots,\alpha_n} a^{(\alpha_1,\alpha_2,\dots,\alpha_n)}_\nu \cdot x^{(1)}_{\alpha_1} \cdot x^{(2)}_{\alpha_2} \cdot \,\cdots\, \cdot x^{(n)}_{\alpha_n} \Big) \cdot e_\nu \,\in W\,.$$

Thus, $\dim \mathrm{Hom}(V_1, V_2, \dots, V_n; W) = \dim W \cdot \prod_\nu \dim V_\nu$.

EXERCISE 4.1. Check that A) a collection of vectors $v_1, v_2, \dots, v_n \in V_1 \times V_2 \times \dots \times V_n$ does not contain the zero vector if and only if there exists a multilinear map φ such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$ B) for a linear $F : U \to W$ and multilinear $\varphi : V_1 \times V_2 \times \dots \times V_n \to U$, the composition $F \circ \varphi : V_1 \times V_2 \times \dots \times V_n \to W$ is multilinear.

4.1.1 Tensor product of vector spaces. Given a multilinear map

$$\tau : V_1 \times V_2 \times \dots \times V_n \to U \tag{4-1}$$

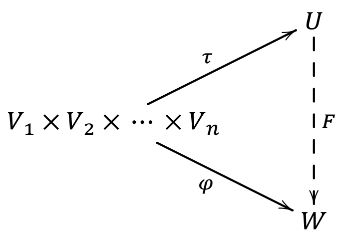
and a vector space W, composing τ with linear maps $F : U \to W$ assigns the map

$$\mathrm{Hom}(U, W) \xrightarrow{F \mapsto F \circ \tau} \mathrm{Hom}(V_1, V_2, \dots, V_n; W) \tag{4-2}$$

which is obviously linear in F.

DEFINITION 4.1

A multilinear map (4-1) is called *universal* if for any vector space W, the linear map (4-2) is an isomorphism. In the expanded form, this means that for every vector space W and multilinear map $\varphi : V_1 \times V_2 \times \dots \times V_n \to W$, there exist a unique linear operator $F : U \to W$ such that $\varphi = F \circ \tau$, i.e., two solid multilinear arrows in the diagram



[1] The usual matrices of dimension 2 and size $d \times m$ describe *linear* maps $V \to W$.

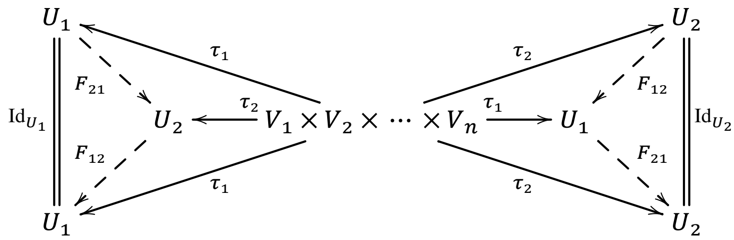
are uniquely completed to a commutative triangle by the dashed linear arrow.

LEMMA 4.1
For every two universal multilinear maps

$$\tau_1 : V_1 \times V_2 \times \cdots \times V_n \to U_1 , \quad \tau_2 : V_1 \times V_2 \times \cdots \times V_n \to U_2 ,$$

there exists a unique linear isomorphism $\iota : U_1 \xrightarrow{\sim} U_2$ such that $\tau_2 = \iota\tau_1$.

PROOF. By the universal properties of τ_1, τ_2, there exists a unique pair of linear maps $F_{21} : U_1 \to U_2$ and $F_{12} : U_2 \to U_1$ that fit in the commutative diagram



Since the factorizations $\tau_1 = \varphi \circ \tau_1$, $\tau_2 = \psi \circ \tau_2$ are unique and hold for $\varphi = \mathrm{Id}_{U_1}$, $\psi = \mathrm{Id}_{U_2}$, we conclude that $F_{21}F_{12} = \mathrm{Id}_{U_2}$ and $F_{12}F_{21} = \mathrm{Id}_{U_1}$. □

LEMMA 4.2
Given a basis $e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, write $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for the vector space with basis formed by $\prod d_i$ formal expressions

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \dots \otimes e_{\alpha_n}^{(n)} , \quad 1 \leqslant \alpha_i \leqslant d_i . \tag{4-3}$$

Then the multilinear map $\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$ sending every collection of basis vectors $\left(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots , e_{\alpha_n}^{(n)}\right) \in V_1 \times V_2 \times \cdots \times V_n$ to the expression (4-3) is universal.

PROOF. For a multilinear $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ and linear $F : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W$, the identity $\varphi = F \circ \tau$ mans exactly that $F\left(e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \dots \otimes e_{\alpha_n}^{(n)}\right) = \varphi\left(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots , e_{\alpha_n}^{(n)}\right)$ for all collections of basis vectors. □

DEFINITION 4.2
The universal multilinear map (4-1) is denoted by

$$\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n , \quad (v_1, v_2, \dots , v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n \tag{4-4}$$

and called *tensor multiplication*. The target space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called the *tensor product* of spaces $V_1, V_2, \dots , V_n$ and its elements are called *tensors*.

4.1.2 Decomposable tensors and Segre varieties. The image of tensor multiplication (4-4) consists of the tensor products $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ called *tensor monomials* or *decomposable tensors*. They do not form a vector space, because the map (4-4) is not linear but multilinear. However, the linear span of decomposable tensors is the whole space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Over an infinite ground field, a random tensor is most likely an indecomposable linear combination of tensor monomials.

Geometrically, the tensor multiplication assigns a map

$$s : \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \to \mathbb{P}\left(V_1 \otimes V_2 \otimes \cdots \otimes V_n\right) \tag{4-5}$$

sending a collection of dimension 1 subspaces $\Bbbk \cdot v_i \subset V_i$ spanned by non zero vectors $v_i \in V_i$ to the dimension 1 subspace $\Bbbk \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_n \subset V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

EXERCISE 4.2. Verify that the map (4-5) is a well defined and injective.

The map (4-5) is called the *Segre embedding* and its image, i.e., the projectivization of the set of decomposable tensors, is called the *Segre variety*. Since the decomposable tensors linearly span the whole space, the Segre variety is not contained in a hyperplane. Note that the dimension of Segre variety equals $\sum m_i$, where $m_i = d_i - 1$, and is much smaller then $\dim \mathbb{P}\left(V_1 \otimes V_2 \otimes \cdots \otimes V_n\right) = \prod(1+m_i)-1$. By the construction, the Segre variety is ruled by n families of projective subspaces of dimensions $m_1, m_2, \dots, m_n$. The simplest example of the Segre variety is provided by the Segre quadric from n° 2.5.1 on p. 22.

EXAMPLE 4.1 (DECOMPOSABLE LINEAR MAPS)
For any two vector spaces U, W, the bilinear map $U^* \times W \to \mathrm{Hom}(U, V)$ is provided by sending $(\xi, w) \in U^* \times W$ to the linear operator $U \to W$, $u \mapsto \langle\, \xi \,,\, u \,\rangle \cdot w$. By the universal property of tensor multiplication, there exists a unique linear map

$$U^* \otimes V \to \mathrm{Hom}(U, V) \tag{4-6}$$

sending every decomposable tensor $\xi \otimes w$ to the same operator. Note that this operator has rank 1, its image is spanned by $w \in W$, and the kernel is $\mathrm{Ann}(\xi) \subset U$.

EXERCISE 4.3. Check that A) every linear map $F : U \to W$ of rank 1 equals $\xi \otimes w$ for appropriate $\xi \in U^*$, $w \in W$ uniquely up to proportionality determined by F B) the linear map (4-6) is an isomorphism for any vector spaces U and V of finite dimensions.

Geometrically, the operators of rank 1 form the Segre variety $S \subset \mathbb{P}_{mn-1} = \mathbb{P}(\mathrm{Hom}(U, W))$, which is ruled by two families of projective spaces $\xi \otimes \mathbb{P}(W)$, $\mathbb{P}(U^*) \otimes w$ and is not contained in a hyperplane. If we fix some bases in U, W, write operators $U \to W$ by their matrices $A = \left(a_{ij}\right)$ in these bases, and use the matrix elements a_{ij} as the homogeneous coordinates in $\mathbb{P}(\mathrm{Hom}(V, W))$, then the Segre variety is described by the equation $\mathrm{rk}\, A = 1$, which encodes the system of homogeneous quadratic equations

$$\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{\ell j} a_{\ell k} - a_{ik} a_{\ell j} = 0$$

for all $1 \leqslant i < \ell \leqslant \dim W$, $1 \leqslant j < k \leqslant \dim U$. The Segre embedding

$$\mathbb{P}(U^*) \times \mathbb{P}(V) = \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \hookrightarrow \mathbb{P}_{mn-1} = \mathbb{P}(\mathrm{Hom}(U, W))$$

takes a pair of points $x = (x_1 : x_2 : \cdots : x_n)$, $y = (y_1 : y_2 : \cdots : y_n)$ to the rank 1 matrix $A(x, y) = y^t \cdot x$ whose $a_{ij} = x_j y_i$. For $\dim U = \dim W = 2$, we get the Segre quadric in $\mathbb{P}_3$ discussed in n° 2.5.1 on p. 22.

4.2 Tensor algebra and contractions. Given a vector space V, we write $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ for the tensor product of n copies of V an call it the n th *tensor power* of V. We also put $V^{\otimes 0} \stackrel{\text{def}}{=} \Bbbk$, $V^{\otimes 1} \stackrel{\text{def}}{=} V$. The infinite direct sum $\mathsf{T}V \stackrel{\text{def}}{=} \bigoplus_{n \geqslant 0} V^{\otimes n}$ is called the *tensor algebra* of V. This is an associative (non-commutative) graded algebra with the multiplication provided by the tensor product of vectors. For every basis $e_1, e_2, \dots, e_n$ in V, the tensor monomials

$$e_{\nu_1} \otimes e_{\nu_2} \otimes \cdots \otimes e_{\nu_m} \tag{4-7}$$

form a basis of $\mathsf{T}V$ over $\Bbbk$. These monomials are multiplied just by writing them sequentially with the sign $\otimes$ between then. Linear combinations of monomials are multiplied by the usual distributivity rules. Thus, $\mathsf{T}V$ may be thought of as the algebra of polynomials in n non-commuting variables e_ν. Another name for $\mathsf{T}V$ is the *free associative* $\Bbbk$*-algebra with unit* spanned by the vector space V. This name emphasizes the following universal property of the $\Bbbk$-linear map

$$\iota : V \hookrightarrow \mathsf{T}V \tag{4-8}$$

embedding V into $\mathsf{T}V$ as the subspace $V^{\otimes 1}$ of linear homogeneous polynomials.

EXERCISE 4.4. Prove that for every associative $\Bbbk$-algebra A with unit and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of associative $\Bbbk$-algebras $\alpha : \mathsf{T}V \to A$ such that[1] $f = \alpha \circ \iota$. Convince yourself that this property characterizes the inclusion (4-8) uniquely up to a unique isomorphism of the target space commuting with the inclusion.

4.2.1 Total contraction and duality. There is the canonical pairing between $(V^*)^{\otimes n}$ and $V^{\otimes n}$ provided by the *total contraction*, which sends $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$, $v = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ to

$$\langle \xi , v \rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n} \langle \xi_i , v_i \rangle . \tag{4-9}$$

Since the right hand side is multilinear in v_i's, every collection of ξ_i's assigns the well defined linear map $V^{\otimes n} \to \Bbbk$, which depends on ξ_i's also multilinearly. Hence, the contraction of decomposable tensors (4-9) is uniquely extended to the bilinear pairing $V^{*\otimes n} \times V^{\otimes n} \to \Bbbk$. For a pair of dual bases $e_1, e_2, \dots, e_n \in V$, $x_1, x_2, \dots, x_n \in V^*$, the tensor monomials $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ and $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_s}$ form the dual bases of $\mathsf{T}V$ and $\mathsf{T}V^*$ with respect to this pairing. In particular, for a finite dimensional vector space V, we have the canonical isomorphism

$$\left(V^{\otimes n}\right)^* \simeq (V^*)^{\otimes n} . \tag{4-10}$$

It follows from the universal property of $V^{\otimes n}$ that the space $\left(V^{\otimes n}\right)^*$ of the linear maps $V^{\otimes n} \to \Bbbk$ is canonically isomorphic to the space of multilinear maps $V \times V \times \cdots \times V \to \Bbbk$, i.e.,

$$\left(V^{\otimes n}\right)^* \simeq \mathrm{Hom}(V, \dots, V ; \Bbbk) . \tag{4-11}$$

Combining (4-10) and (4-11) leads to the canonical isomorphism

$$(V^*)^{\otimes n} \simeq \mathrm{Hom}(V, \dots, V ; \Bbbk) . \tag{4-12}$$

It sends a decomposable tensor $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$ to the multilinear map $V \times V \times \cdots \times V \to \Bbbk$ taking $(v_1, v_2, \dots, v_n) \mapsto \prod_{i=1}^n \xi_i(v_i)$.

[1] In other words, for every $\Bbbk$-algebra A, the homomorphisms of $\Bbbk$-algebras $\mathsf{T}V \to A$ stay in bijection with the $\Bbbk$-linear maps $V \to A$.

4.2.2 Partial contractions. Consider two inclusions[1] of sets

$$\{1,2,\ \dots\ ,p\} \xleftarrow{I} \{1,2,\ \dots\ ,m\} \xrightarrow{J} \{1,2,\ \dots\ ,q\}\,,$$

and write i_ν, j_ν for $I(\nu)$, $J(\nu)$ respectively. Thus, we have two numbered collections of indexes $I=(i_1,i_2,\dots,i_m)$, $J=(j_1,j_2,\dots,j_m)$ staying in the fixed bijection. A *partial contraction* of $V^{*\otimes p}$ and $V^{\otimes q}$ in indexes I, J is the linear map

$$c^I_J : V^{*\otimes p}\otimes V^{\otimes q} \to V^{*\otimes(p-m)}\otimes V^{\otimes(q-m)}$$

which contracts i_ν th factor of $V^{*\otimes p}$ with j_ν th factor of $V^{\otimes q}$ for every $\nu=1,\,2,\,\dots\,,m$ and keeps all the other factors in their initial order:

$$\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_p\otimes v_1\otimes v_2\otimes\cdots\otimes v_q \mapsto \prod\nolimits_{\nu=1}^m \langle\,\xi_{i_\nu}\,,\,v_{j_\nu}\,\rangle\cdot(\bigotimes_{i\notin I}\xi_i)\otimes(\bigotimes_{j\notin J}v_j)\,. \tag{4-13}$$

Note that different choices of the maps I, J lead to the different contraction maps even if the images of I, J remain unchanged.

Example 4.2 (innner product between vectors and multilinear forms)
Let us treat a n-linear form $\varphi(v_1,v_2,\dots,v_n)$ as a tensor from $V^{*\otimes n}$ via isomorphism (4-12). The contraction of this tensor with a vector $v\in V$ in the first tensor factor is a tensor from $V^{*\otimes(n-1)}$, which can be considered as an $(n-1)$-linear form on V. This form is called the *innner product* of v and φ and denoted by $i_v\varphi$ or $v\llcorner\varphi$.

Exercise 4.5. Check that $i_v\varphi(w_1,w_2,\dots,w_{n-1})=\varphi(v,w_1,w_2,\dots,w_{n-1})$.

4.2.3 The linear support of a tensor. Given a tensor $t\in V^{\otimes n}$, the intersection of all vector subspaces $W\subset V$ such that $t\in W^{\otimes n}$ is called the *linear support* of t and denoted by $\mathrm{Supp}(t)\subset V$. It follows from the next Exercise 4.6 that $\mathrm{Supp}(t)$ is the unique minimal[2] subspace in V among those $W\subset V$ for which $t\in W^{\otimes n}$.

Exercise 4.6. For any subspaces $U,W\subset V$, verify that $U^{\otimes n}\cap W^{\otimes n}=(U\cap W)^{\otimes n}$ in $V^{\otimes n}$.

The dimension of $\mathrm{Supp}\,t$ is called the *rank* of t and denoted by $\mathrm{rk}\,t \overset{\text{def}}{=} \dim\mathrm{Supp}\,t$. We say that t is *degenerated* if $\mathrm{rk}\,t<\dim V$. In this case, the number of variables in the expansion of t through the basis tensor monomials can be reduced by a linear change of variables.

Exercise 4.7. Show that if $\dim\mathrm{Supp}(t)=1$ and the ground field is algebraically closed, then $t=\lambda\cdot v^{\otimes n}$ for some $\lambda\in\Bbbk$, $v\in V$.

The space $\mathrm{Supp}(t)$ admits an effective description as a linear span of some finite collection of vectors constructed by means of contraction maps. Namely, for every injective[3] map

$$J:\{1,\,2,\,\dots\,,(n-1)\}\hookrightarrow\{1,\,2,\,\dots\,,n\}\,, \tag{4-14}$$

write $\{j_1,j_2,\dots,j_{n-1}\}\subset\{1,2,\,\dots\,,n\}$ for the image of J and $\hat{\jmath}$ for the remaining index outside $\mathrm{im}\,J$. Consider the contraction map

$$c^J_t : V^{*\otimes(n-1)}\to V\,,\quad \xi\mapsto c^{(1,\,2,\,\dots\,,(n-1))}_{(j_1,j_2,\dots,j_{n-1})}(\xi\otimes t) \tag{4-15}$$

[1] Not necessary monotonous.
[2] With respect to inclusions.
[3] Not necessary monotonous.

which couples ν th tensor factor of $V^{*\otimes(n-1)}$ with j_ν th tensor factor of t for all $1 \leqslant \nu \leqslant (n-1)$. The result of such contraction is obviously a linear combination of $\hat{j}$ th tensor factors of t. Thus, it belongs to $\operatorname{Supp}(t)$.

THEOREM 4.1
For every $t \in V^{\otimes n}$, the linear support $\operatorname{Supp}(t) \subset V$ is spanned by the images of all contraction maps (4-15) coming from $n!$ different choices of the map (4-14).

PROOF. Let $\operatorname{Supp}(t) = W \subset V$. It is enough to check that every linear form $\xi \in V^*$ annihilating all the subspaces $\operatorname{im}\left(c_t^J\right)$ annihilates W as well. Assume the contrary: let a linear form $\xi \in V^*$ annihilate all $c_t^J\left(V^{*\otimes(n-1)}\right)$ but have a non-zero restriction on W. Chose a basis $\xi_1, \xi_2, \dots, \xi_d \in V^*$ such that $\xi_1 = \xi$ and the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ on W form a basis in W^*. Expand t through the tensor monomials built from the dual basis vectors $w_1, w_2, \dots, w_k \in W$. The value

$$\xi\left(c_t^J(\xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}})\right)$$

is equal to the complete contraction of t with the basic monomial $\xi_1 \otimes \xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}}$ in the order of coupling prescribed by J. This contraction kills all tensor monomials in the expansion of t except for the one, dual to the monomial obtained from $\xi_1 \otimes \xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}}$ by some permutation of factors depending on J. Thus, the result of contraction is equal to the coefficient of some monomial containing w_1 in the expansion of t. Since every such monomial can be reached by appropriate choice of J, we conclude that $w_1 \notin \operatorname{Supp}(t)$. Contradiction. □

4.3 Symmetric and grassmannian algebras. A multilinear map $\varphi : V \times V \times \cdots \times V \to U$ is called *symmetric* if it remains unchanged under permutations of the arguments, and *alternating* if it vanishes as soon some of the arguments coincide.

EXERCISE 4.8. Verify that under a permutation of the arguments, the value of an alternating multilinear map is multiplied by the sign of permutation. Convince yourself that this property implies the alternating property if $\operatorname{char} \Bbbk \neq 2$.

We write $\operatorname{Sym}^n(V, U) \subset \operatorname{Hom}(V, \dots, V; U)$ and $\operatorname{Alt}^n(V, U) \subset \operatorname{Hom}(V, \dots, V; U)$ for subspaces of symmetric and alternating multilinear maps. Everything said about the universal multilinear maps in n° 4.1.1 on p. 38 makes sense separately for the symmetric and alternating maps as well. The universal symmetric multilinear map is denoted by

$$\sigma : V \times V \times \cdots \times V \to S^n V, \quad (v_1, v_2, \dots, v_n) \mapsto v_1 v_2 \dots v_n, \tag{4-16}$$

and called the *commutative* multiplication of vectors. Its target space $S^n V$ is called the n th *symmetric power* of V. The universal alternating multilinear map is denoted by

$$\alpha : V \times V \times \cdots \times V \to \Lambda^n V, \quad (v_1, v_2, \dots, v_n) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_n, \tag{4-17}$$

and called the *exterior*[1] multiplication of vectors. Its target space $\Lambda^n V$ is called the n th *exterior power* of V. The universal symmetric and alternating multilinear maps are unique up to a unique isomorphism of the target space commuting with the universal map. The both can be constructed for all n at once by factorizing the tensor algebra $\mathsf{T}V$ by appropriate two-sided ideals.

[1] Also known as *grassmannian* or *super-commutative*.

4.3.1 The symmetric algebra. Write $I_{\text{com}} \subset \mathsf{T}V$ for a two-sided ideal spanned by all the differences

$$u \otimes w - w \otimes u\,, \quad u, w \in V\,. \tag{4-18}$$

This ideal is obviously homogeneous in the sense that $I_{\text{com}} = \bigoplus_{n\geqslant 0}\left(I_{\text{com}} \cap V^{\otimes n}\right)$, and the degree n component $I_{\text{com}} \cap V^{\otimes n}$ of I_{com} is linearly generated over $\Bbbk$ by all differences of the form

$$(\cdots \otimes v \otimes w \otimes \cdots) - (\cdots \otimes w \otimes v \otimes \cdots)\,, \tag{4-19}$$

where the both terms are decomposable of degree n and vary only in the order of v, w. The factor algebra $SV \overset{\text{def}}{=} \mathsf{T}V/I_{\text{com}}$ is called the *symmetric algebra* of V. The multiplication in SV comes from the tensor multiplication in $\mathsf{T}V$ and is commutative, because of the relations $uw = wu$ appearing after the factorization through (4-18). The symmetric algebra is graded

$$SV = \bigoplus_{n\geqslant 0} S^nV\,, \quad \text{where } S^nV \overset{\text{def}}{=} V^{\otimes n}/(I_{\text{com}} \cap V^{\otimes n})\,.$$

EXERCISE 4.9. Show that for every basis $e_1, e_2, \dots, e_d \subset V$, the monomials $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ form a basis of SV over $\Bbbk$.

Thus, we get an isomorphism of algebras $SV \simeq \Bbbk[e_1, e_2, \dots, e_d]$. Under this isomorphism, S^nV turns to the subspace of homogeneous polynomials of degree n.

EXERCISE 4.10. Deduce from the universal property of tensor multiplication that the map

$$V \times V \times \cdots \times V \to S^nV$$

provided by the multiplication in SV is the universal symmetric multilinear map. Convince yourself that SV is the *free commutative* $\Bbbk$*-algebra* spanned by V in the sense that for every commutative $\Bbbk$-algebra A and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : SV \to A$ such that $f = \widetilde{\varphi} \circ \iota$, where $\iota : V \hookrightarrow SV$ embeds V in SV as the space of linear homogeneous polynomials. Show that the latter embedding is uniquely characterized by the previous universal property up to a unique isomorphism commuting with ι.

4.3.2 The exterior[1] **algebra** of a vector space V is defined as the factor algebra $\Lambda V \overset{\text{def}}{=} \mathsf{T}V/I_{\text{alt}}$, where $I_{\text{alt}} \subset \mathsf{T}V$ is the two-sided ideal generated by all tensor squares $v \otimes v$, $v \in V$.

EXERCISE 4.11. Check that the space $I_{\text{alt}} \cap V^{\otimes 2}$ contains all sums $v \otimes w + w \otimes v$, $v, w \in V$, and is linearly generated over $\Bbbk$ by these sums if $\operatorname{char}\Bbbk \neq 2$.

The ideal I_{alt} also splits in the direct sum of homogeneous components

$$I_{\text{alt}} = \bigoplus_{n\geqslant 0}\left(I_{\text{alt}} \cap V^{\otimes n}\right)\,.$$

The degree n component $I_{\text{alt}} \cap V^{\otimes n}$ is spanned by decomposable tensors of the form

$$(\cdots \otimes v \otimes v \otimes \cdots)\,, \quad v \in V\,.$$

By Exercise 4.11, all the sums $(\cdots \otimes v \otimes w \otimes \cdots) + (\cdots \otimes w \otimes v \otimes \cdots)$ belong to $I_{\text{alt}} \cap V^{\otimes n}$ as well and linearly generate it over $\Bbbk$ as soon $\operatorname{char}\Bbbk \neq 2$. The multiplication in ΛV is called the

[1] Also known as the *grassmannian algebra* or *free super-commutative algebra* of V.

exterior[1] *multiplication* and denoted by the wedge sign $\wedge$. Note that for any $u, w \in V$, the relations $u \wedge u = 0$ and $u \wedge w = -w \wedge u$ hold in $\Lambda^2 V$. Hence, under a permutation of factors, the exterior product of vectors is multiplied by the sign of permutation:

$$\forall\, g \in S_k \quad v_1 \wedge v_2 \wedge \cdots \wedge v_k = \operatorname{sgn}(g) \cdot v_{g_1} \wedge v_{g_2} \wedge \cdots \wedge v_{g_k}\,.$$

This property of a multiplication is known as the *super-commutativity*. Like the symmetric algebra, the exterior algebra is graded:

$$\Lambda V = \bigoplus_{n \geqslant 0} \Lambda^n V\,, \quad \text{where } \Lambda^n V \overset{\text{def}}{=} V^{\otimes n} / (I_{\text{alt}} \cap V^{\otimes n})\,.$$

EXERCISE 4.12. Deduce from the universal property of tensor multiplication that the map

$$V \times V \times \cdots \times V \to \Lambda^n V$$

provided by the exterior multiplication in ΛV is the universal alternating multilinear map. Convince yourself that ΛV is the *free super-commutative* $\Bbbk$*-algebra* spanned by V in the sense that for every super-commutative $\Bbbk$-algebra A and $\Bbbk$-linear map $f : V \to A$, there exists a unique homomorphism of $\Bbbk$-algebras $\widetilde{f} : SV \to A$ such that $f = \widetilde{\varphi} \circ \iota$, where $\iota : V \hookrightarrow SV$ embeds V in ΛV as the subspace $\Lambda^1 V = V^{\otimes 1}$. Show that the latter embedding is uniquely characterized by the previous universal property up to a unique isomorphism commuting with ι.

PROPOSITION 4.1
For every basis $e_1, e_2, \dots, e_d$ in V the grassmannian monomials $e_I \overset{\text{def}}{=} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, numbered by strictly increasing multi-indexes $I = (i_1, i_2, \dots, i_n)$, $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$, form a basis of $\Lambda^n V$.

PROOF. Write U for the vector space of dimension $\binom{d}{n}$ with the basis formed by symbols ξ_I, where $I = (i_1, i_2, \dots, i_n)$ runs through all strictly increasing sequences of length n in $1, 2, \dots, d$. Consider the multilinear map $\alpha : V \times V \times \cdots \times V \to U$ that takes an arbitrary collection $e_{j_1}, e_{j_2}, \dots, e_{j_n}$ of the basis vectors from V to $\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n}) = \operatorname{sgn}(\sigma) \cdot \xi_I$, where $I = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(n)})$ is the strictly increasing permutation of the indexes $j_1, j_2, \dots, j_n$ and we put $\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n}) = 0$ when some of j_ν's coincide. For any alternating multilinear map $\varphi : V \times V \times \cdots \times V \to W$, there exists a unique linear operator $F : U \to W$ such that $\varphi = F \circ \alpha$: the action F on the basis of U has to be $F(\xi_{(i_1, i_2, \dots, i_n)}) = \varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n})$. Thus, α is the universal alternating multilinear map. Hence, there exists an isomorphism $U \xrightarrow{\sim} \Lambda^n V$ sending $\xi_I \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. □

COROLLARY 4.1
$\dim \Lambda^n V = \binom{d}{n}$, where $d = \dim V$. In particular, $\Lambda^n V = 0$ for $n > d$, and $\dim \Lambda V = 2^d$.

EXERCISE 4.13. Check that $\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha$ for any $\alpha \in \Lambda^a V$, $\beta \in \Lambda^b V$, and describe the centre[2] $Z(\Lambda V)$.

[1] Or *grassmannian*, or *super-commutative*
[2] That is, all elements commuting with every element of the algebra.

4.3.3 Grassmannian polynomials. It follows from Proposition 4.1 that every choice of basis $e_1, e_2, \dots, e_d$ in a vector space V assigns the isomorphism of $\Bbbk$-algebras

$$\Lambda V \xrightarrow{\sim} k\langle e_1, e_2, \dots, e_d\rangle ,$$

where $k\langle e_1, e_2, \dots, e_d\rangle$ stays for the algebra of *grassmannian polynomials*, i.e., polynomials with coefficients from $\Bbbk$ in the variables e_i satisfying the relations $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. When work with the grassmannian polynomials, we always write $I = (i_1, i_2, \dots, i_n)$ for a strictly increasing collection of indexes, $\hat{I} = (\hat{\imath}_1, \hat{\imath}_2, \dots, \hat{\imath}_{d-n}) = \{1, 2, \dots, d\} \smallsetminus I$ for the complementary strictly increasing collection, and $\#I \overset{\text{def}}{=} n$ for the *length* of I. The sum $|I| \overset{\text{def}}{=} \sum_\nu i_\nu$ is called the *weight* of I.

EXERCISE 4.14. Check that $e_I \wedge e_{\hat{I}} = (-1)^{|I|+\frac{1}{2}\#I(1+\#I)} \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_d$.

EXAMPLE 4.3 (LINEAR SUBSTITUTION OF VARIABLES)
Let the variables $e_1, e_2, \dots, e_n$ be linearly expressed through the variables $\xi_1, \xi_2, \dots, \xi_m$ as

$$e_i = \sum_j a_{ij}\,\xi_j \tag{4-20}$$

for some $n \times m$ matrix $A = \left(a_{ij}\right)$. Then the grassmannian monomials e_I are expressed through ξ_I as

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = \Big(\sum_{j_1} a_{i_1 j_1}\,\xi_{j_1}\Big) \wedge \Big(\sum_{j_2} a_{i_2 j_2}\,\xi_{j_2}\Big) \wedge \cdots \wedge \Big(\sum_{j_n} a_{i_n j_n}\,\xi_{j_n}\Big) =$$
$$= \sum_{1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant n}\ \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{i_1 j_{\sigma(1)}} a_{i_2 j_{\sigma(2)}} \cdots a_{i_n j_{\sigma(n)}}\, \xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n} = \sum_J a_{IJ}\,\xi_J ,$$

where J runs through increasing collections of length n and a_{IJ} denotes the $n \times n$ minor of A situated in the rows $i_1, i_2, \dots, i_n$ and columns $j_1, j_2, \dots, j_n$.

EXAMPLE 4.4 (MULTIROW COFACTOR EXPANSIONS OF DETERMINANT)
Let us perform the substitution (4-20) in the identity from Exercise 4.14 using a square $d \times d$ matrix A. The left hand side of the identity turns to

$$\Big(\sum_{\substack{K:\\ \#K=\#I}} a_{IK}\,\xi_K\Big) \wedge \Big(\sum_{\substack{L:\\ \#L=(d-\#I)}} a_{\hat{I}L}\,\xi_L\Big) = (-1)^{\frac{1}{2}\#I(1+\#I)} \sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|} a_{IK} a_{\hat{I}\hat{K}}\, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d .$$

The right hand side becomes $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|}\det(a_{ij}) \cdot \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d$. Thus, for every collection $I = (i_1, i_2, \dots, i_n)$ of rows in a square matrix $A = (a_{ij})$, the following relation holds

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} a_{\hat{I}\hat{K}} = \det(a_{ij}) , \tag{4-21}$$

where the summation goes over all $n \times n$ minors a_{IK} situated in the rows $(i_1, i_2, \dots, i_n)$.

If we replace $\hat{I}$ by another collection $\hat{J}$ complementary to the other $J \neq I$, then we get in the right hand side $e_I \wedge e_{\hat{J}} = 0$. Thus, for every $J \neq I$,

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} a_{\hat{J}\hat{K}} = 0 . \tag{4-22}$$

The identities (4-21) and (4-22) are known as the *Laplace relations*. They generalize the cofactor expansions of determinants. If we organize $n\times n$ minors of A and their complements in two $\binom{d}{n}\times\binom{d}{n}$ matrices $\mathcal{A}_n = \left(a_{IJ}\right)$ and $\mathcal{A}_n^\vee = \left(a_{IJ}^\vee\right)$, where[1] $a_{IJ}^\vee = (-1)^{|I|+|J|}a_{\hat{J}\hat{I}}$, then all the Laplace relations can be combined in the one matrix identity $\mathcal{A}_n \cdot \mathcal{A}_n^\vee = \det A \cdot E$.

EXERCISE 4.15. Write the Laplace relations for multicolumn cofactor expansions and prove that $\mathcal{A}_n^\vee \cdot \mathcal{A}_n = \det A \cdot E$ as well.

EXAMPLE 4.5 (REDUCTION OF GRASSMANNIAN QUADRATIC FORM)
Certainly, a grassmannian quadratic form can not be reduced to a «sum of squares» like in Proposition 2.1 on p. 16. However, every homogeneous grassmannian polynomial of degree two over an arbitrary field $\Bbbk$ takes in appropriate coordinates the form

$$\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots + \xi_{2r-1} \wedge \xi_{2r}, \tag{4-23}$$

called *the Darboux normal form*. To achieve it for a given $\omega \in \Lambda^2 V$, we renumber the initial basis $e_1, e_2, \dots, e_n$ of V in such a way that $\omega = e_1 \wedge (\alpha_2 e_2 + \cdots + \alpha_n e_n) + e_2 \wedge (\beta_3 e_3 + \cdots + \beta_n e_n) +$ (terms without e_1, e_2), where $\alpha_2 \neq 0$. Then we pass to the new basis $\{e_1, \xi_2, e_3, \dots, e_n\}$ which has $\xi_2 = \alpha_2 e_2 + \cdots + \alpha_n e_n$. The substitution $e_2 = \left(\xi_2 - \beta_3 e_3 - \cdots - \beta_n e_n\right)/\alpha_2$ in ω leads to

$$\begin{aligned}\omega = e_1 \wedge \xi_2 + \xi_2 \wedge (\gamma_3 e_3 + \cdots + \gamma_n e_n) + (\text{terms without } \xi_2) = \\ = (e_1 - \gamma_3 e_3 - \cdots - \gamma_n e_n) \wedge \xi_2 + (\text{terms without } e_1, \xi_2)\,.\end{aligned}$$

Now we pass to the basis $\{\xi_1, \xi_2, e_3, \dots, e_n\}$, where $\xi_1 = e_1 - \gamma_3 e_3 - \cdots - \gamma_n e_n$. In this basis,

$$\omega = \xi_1 \wedge \xi_2 + (\text{terms without } \xi_1, \xi_2)$$

and we can continue by induction.

CONVENTION 4.1. In the rest of §4 we assume on default that $\operatorname{char}(\Bbbk) = 0$.

4.4 Symmetric and alternating tensors. The symmetric group S_n acts on $V^{\otimes n}$ by permutations of factors in decomposable tensors: for $g \in S_n$, we put

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{g(1)} \otimes v_{g(2)} \otimes \cdots \otimes v_{g(n)}\,. \tag{4-24}$$

Since the right hand side is multilinear in $v_1, v_2, \dots, v_n$, this formula assigns the well defined linear map $g : V^{\otimes n} \to V^{\otimes n}$.

DEFINITION 4.3
A tensor $t \in V^{\otimes n}$ is called *symmetric*, if $g(t) = t$ for all $g \in S_n$. A tensor $t \in V^{\otimes n}$ is called *alternating*, if $g(t) = \operatorname{sgn}(g) \cdot t$ for all $g \in S_n$. We write $\operatorname{Sym}^n V = \{\, t \in V^{\otimes n} \mid \forall g \in S_n\ \sigma(t) = t\}$ and $\operatorname{Alt}^n V = \{\, t \in V^{\otimes n} \mid \forall g \in S_n\ g(t) = \operatorname{sgn}(g)\}$ for the space of symmetric and alternating tensors respectively. Note that both are the *sub*spaces in $V^{\otimes n}$, and they should not be confused with the *quotient* spaces $S^n V$, $\Lambda^n V$ of $V^{\otimes n}$.

[1] Note that I, J swap places.

4.4.1 Standard bases. For every basis $e_1, e_2, \dots, e_d$ in V, a basis of $\mathrm{Sym}^n V$ is formed by the *complete symmetric tensors*

$$e_{[m_1,m_2,\dots,m_d]} \overset{\text{def}}{=} \begin{pmatrix} \text{the sum of all tensor monomials containing} \\ m_1 \text{ factors } e_1,\ m_2 \text{ factors } e_2,\ \dots,\ m_d \text{ factors } e_d\,, \end{pmatrix} \tag{4-25}$$

because all the summands appear in the expansion of every symmetric tensor t with equal coefficients. The tensors (4-25) are indexed by the collections of non-negative integers $(m_1, m_2, \dots, m_d)$ such that $\sum_\nu m_\nu = n$.

EXERCISE 4.16. Make it sure that the sum (4-25) consists of $\frac{n!}{m_1!\,m_2!\cdots m_d!}$ terms.

Similarly, a basis of $\mathrm{Alt}^n V$ is formed by the *complete alternating tensors*

$$e_{\langle i_1,i_2,\dots,i_n\rangle} \overset{\text{def}}{=} \sum_{g\in S_n} \mathrm{sgn}(g)\cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}} \tag{4-26}$$

numbered by increasing sequences $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

4.5 Polarization of commutative polynomials. The quotient map $V^{\otimes n} \twoheadrightarrow S^n V$ sends every summand of (4-25) to the same commutative monomial $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. Thus, this map sends $e_{[m_1,m_2,\dots,m_d]}$ to $\frac{n!}{m_1!\,m_2!\cdots m_d!}\cdot e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. Over the ground field of zero characteristic, we conclude that for every n, the factorization through the commutativity relations assigns the isomorphism $\mathrm{Sym}^n V \xrightarrow{\sim} S^n V$. The inverse isomorphism is denoted by

$$\mathrm{pl} : S^n V \xrightarrow{\sim} \mathrm{Sym}^n V\,, \quad f \mapsto \widetilde{f}\,,$$

and called the *complete polarization* of polynomials. For the dual space V^*, the complete polarization map $\mathrm{pl} : S^n V^* \xrightarrow{\sim} \mathrm{Sym}^n V^*$ sends every monomial $f = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$ to the tensor $\widetilde{f} = \frac{m_1!\,m_2!\cdots m_d!}{n!}\cdot x_{[m_1,m_2,\dots,m_d]} \in \mathrm{Sym}^n V^*$, which can be viewed as the symmetric multilinear map $\widetilde{f} : V \times V \times \dots \times V \to \Bbbk$ acting on a collection of vectors $v_1, v_2, \dots, v_n \in V \times V \cdots \times V$ via the complete contraction with $v_1 \otimes v_2 \otimes \cdots \otimes v_n$.

EXERCISE 4.17. Verify that for every $v \in V$, the complete contraction of $v^{\otimes n}$ with

$$\frac{m_1!\; m_2! \cdots m_d!}{n!}\cdot x_{[m_1,m_2,\dots,m_d]}$$

is equal to the result of evaluation of monomial $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \in \Bbbk[x_1, x_2, \dots, x_n]$ on the coordinates of v.

We conclude that the polynomial function $f : \mathbb{A}(V) \to \Bbbk$ attached to a homogeneous polynomial $f \in S^n V$ in n° 1.1.2 on p. 3 is described in coordinate-free terms as $f(v) = \widetilde{f}(v, v, \dots, v)$, where $\widetilde{f} \in \mathrm{Sym}^n V^* \subset V^{*\otimes n}$ is the unique symmetric tensor mapped to f under factorization through the commutativity relations and considered as a symmetric multilinear map $V \times V \times \cdots \times V \to \Bbbk$. For $n = 2$, we get the polarization of quadratic forms considered in n° 2.1.1 on p. 16.

Since the value $\widetilde{f}(v_1, v_2, \dots, v_n)$ does not depend on the order of arguments, we write

$$\widetilde{f}\left(w_1^{k_1}, w_2^{k_2}, \dots, w_s^{k_s}\right)$$

when the collection $(v_1, v_2, \dots, v_n)$ consists of k_1 vectors w_1, k_2 vectors w_2, … , k_s vectors w_s.

EXERCISE 4.18. For any polynomial $f \in S^n V^*$ and vectors $v_1, v_2, \dots, v_k \in V$, show that

$$f\left(v_1 + v_2 + \cdots + v_k\right) = \widetilde{f}\left(\left(v_1 + v_2 + \cdots + v_k\right)^n\right) = \sum_{m_1 m_2 \dots m_k} \frac{n!}{m_1!\, m_2! \cdots m_k!} \cdot \widetilde{f}\left(v_1^{m_1}, v_2^{m_2}, \dots, v_k^{m_k}\right), \tag{4-27}$$

where the summation goes over all integer $m_1, m_2, \dots, m_k$ such that $m_1 + m_2 + \cdots + m_k = n$ and $0 \leqslant m_\nu \leqslant n$ for all ν.

PROPOSITION 4.2
The complete polarization of a homogeneous polynomial $f \in S^n V^*$ on a vector space[1] V over a field of zero characteristic can be computed by the formula

$$n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \sum_{I \subsetneq \{1,\dots,n\}} (-1)^{\#I} f\Big(\sum_{i \notin I} v_i\Big), \tag{4-28}$$

where the left summation goes over all proper subsets $I \subsetneq \{1, 2, \dots, n\}$, including $I = \varnothing$, for which we put $\#\varnothing = 0$.

EXAMPLE 4.6
For homogeneous quadratic and cubic polynomials $q \in S^2V^*$, $f \in S^3V^*$, we get

$$2\widetilde{q}(u, w) = q(u + w) - q(u) - q(w),$$
$$6\,\widetilde{f}(u, v, w) = f(u + v + w) - f(u + v) - f(u + w) - f(v + w) + f(u) + f(v) + f(w).$$

PROOF OF PROPOSITION 4.2. In the expansion (4-27) for

$$f(v_1 + v_2 + \cdots + v_n) = \widetilde{f}\left((v_1 + v_2 + \cdots + v_n)^n\right),$$

there is just one term containing all the vectors $v_1, v_2, \dots, v_n$, namely $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n)$. For a proper subset $I \subsetneq \{1, 2, \dots, n\}$, every summand which contains no v_i with $i \in I$ appears in (4-27) with the same coefficient as in the expansion (4-27) written for $f(\sum_{i \notin I} v_i)$, because the latter is obtained from $f(v_1 + v_2 + \cdots + v_n)$ by setting $v_i = 0$ for all $i \in I$. Removal of these summands via the standard combinatorial inclusion-exclusion procedure leads to the required formula

$$n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = f\Big(\sum_{\nu} v_\nu\Big) - \sum_{\{i\}} f\Big(\sum_{\nu \neq i} v_\nu\Big) + \sum_{\{i,j\}} f\Big(\sum_{\nu \neq i,j} v_\nu\Big) - \sum_{\{i,j,k\}} f\Big(\sum_{\nu \neq i,j,k} v_\nu\Big) + \cdots.$$

□

[1] Not necessary finite dimensional.

4.5.1 Duality. For a vector space V of finite dimensuon over a field of zero characteristic, the complete contraction between $V^{\otimes m}$ and $V^{*\otimes m}$ provides the spaces $S^m V$ and $S^m V^*$ with the perfect pairing that couples polynomials $f \in S^n V$ and $g \in S^n V^*$ to the complete contraction of their complete polarizations $\widetilde{f} \in V^{\otimes m}$ and $\widetilde{g} \in V^{*\otimes m}$.

EXERCISE 4.19. For a pair of dual bases $e_1, e_2, \dots, e_d \in V$, $x_1, x_2, \dots, x_d \in V^*$, verify that all the non-zero couplings between the basis monomials are exhausted by

$$\langle\, e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}\,,\, x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \,\rangle = \frac{m_1!\, m_2! \cdots m_d!}{n!}\,. \tag{4-29}$$

Note that the monomials constructed from the dual basis vectors become the dual bases of the polynomial rings only after rescaling by appropriate combinatorial factors.

4.5.2 Derivative of a polynomial along a vector. Associated with every vector $v \in V$ is the linear map $i_v : V^{*\otimes n} \to V^{*\otimes(n-1)}$, $\varphi \mapsto i_v\varphi$, provided by the inner multiplication[1] of n-linear forms on V by v, which takes an n-linear form $\varphi \in V^{*\otimes n}$ to the $(n-1)$-linear form

$$i_v\varphi(v_1, v_2, \dots, v_{n-1}) = \varphi(v, v_1, v_2, \dots, v_{n-1})\,.$$

Composing this map with preceded complete polarization$S^n V^* \xrightarrow{\sim} \operatorname{Sym}^n V^* \subset V^{*\otimes n}$ and subsequent factorization $\sigma : V^{*\otimes(n-1)} \twoheadrightarrow S^{n-1} V^*$ through the commutativity relations[2], assigns the linear map

$$\operatorname{pl}_v : S^n V^* \to S^{n-1} V^*\,, \quad f(x) \mapsto \operatorname{pl}_v f(x) \overset{\text{def}}{=} \widetilde{f}(v,\, x,\, x,\, \dots\,,\, x)\,, \tag{4-30}$$

which depends linearly on $v \in V$. This map fits in the commutative diagram

$$\begin{array}{ccc} V^{*\otimes n} \supset \operatorname{Sym}^n V^* & \xrightarrow{\;i_v\;} & V^{*\otimes(n-1)} \\ {\scriptstyle \operatorname{pl}}\uparrow\wr & & \downarrow{\scriptstyle \sigma} \\ S^n V^* & \xrightarrow{\;\operatorname{pl}_v\;} & S^{n-1} V^*\,. \end{array} \tag{4-31}$$

The polynomial $\operatorname{pl}_v f(x) \widetilde{f}(v, x, \dots\, x) \in S^{n-1}(V^*)$ is called the *polar* of v with respect to f. For $n = 2$, the polar of a vector v with respect to a quadratic worm $f \in S^2 V^*$ is the linear form $w \mapsto \widetilde{f}(v, w)$ considered[3] in n° 2.2.1 on p. 18.

In terms of dual bases $e_1, e_2, \dots, e_d \in V$, $x_1, x_2, \dots, x_d \in V^*$, the contraction of the first tensor factor in $V^{*\otimes n}$ with the basis vector $e_i \in V$ maps the complete symmetric tensor $x_{[m_1, m_2, \dots, m_n]}$ either to the complete symmetric tensor containing the $(m_i - 1)$ factors x_i or to zero for $m_i = 0$. Hence, $pl_{e_i} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} = \frac{m_i}{n}\, x_1^{m_1} \dots x_{i-1}^{m_{i-1}} x_i^{m_i - 1} x_{i+1}^{m_{i+1}} \dots x_d^{m_d} = \frac{1}{n}\frac{\partial}{\partial x_i} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$. Since $\operatorname{pl}_v f$ is linear in both v, f, we conclude that for every $v = \sum \alpha_i e_i$, the polar polynomial of v with respect to f is nothing but the *derivative* of the polynomial f along the vector v divided by $\deg f$, i.e.,

$$\operatorname{pl}_v f = \frac{1}{\deg(f)}\, \partial_v f = \frac{1}{\deg(f)} \sum_{i=1}^{d} \alpha_i \frac{\partial f}{\partial x_i}\,.$$

[1]See Example 4.2 on p. 42.

[2]Which is the linear map corresponding to the commutative multiplication of covectors from formula (4-16) on p. 43 by the universal property of tensor product.

[3]Recall that the zero set of this form in $\mathbb{P}(V)$ is the hyperplane intersecting the quadric $V(f) \subset \mathbb{P}(V)$ along its apparent contour viewed from v.

Note that this forces the right hand side to be independent on the choice of dual bases in V and V^*. It follows from the definition of polar map that the derivatives along vectors commute, $\partial_u\partial_w = \partial_w\partial_u$, and for all $u, w \in V$, $f \in S^nV^*$, $0 \leqslant m \leqslant n$, the following relation holds:

$$m!\,\frac{\partial^m f}{\partial u^m}(w) = n!\,\widetilde{f}(u^m, w^n) = (n-m)!\,\frac{\partial^{n-m} f}{\partial w^{n-m}}(u)\,, \tag{4-32}$$

EXERCISE 4.20. Prove *the Leibniz rule* $\partial_v(fg) = \partial_v(f)\cdot g + f\cdot\partial_v(g)$ and show that

$$\widetilde{f}(v_1, v_2, \dots, v_n) = \frac{1}{n!}\partial_{v_1}\partial_{v_2}\,\dots\,\partial_{v_n} f\,.$$

EXAMPLE 4.7 (TAYLOR'S EXPANSION)
For $k = 2$, the expansion (4-27) from Exercise 4.18 turns to the identity

$$f(u+w) = \widetilde{f}(u+w,\, u+w,\, \dots,\, u+w) = \sum_{m=0}^{n}\binom{n}{m}\cdot\widetilde{f}(u^m, w^{n-m}),$$

where $n = \deg f$. It holds for any polynomial $f \in S^nV^*$ and all vectors $u, w \in V$. The relations (4-32) allow us to rewrite this identity as the *Taylor expansion* for f at u:

$$f(u+w) = \sum_{m=0}^{\deg f}\frac{1}{m!}\,\partial_w^m f(u), \tag{4-33}$$

which is an exact equality in the polynomial ring SV^*.

4.5.3 Polars and tangents. Given a hypersurface $S = V(f) \subset \mathbb{P}(V)$ of degree n and a line $\ell = (pq) \subset \mathbb{P}(V)$, the intersection $\ell\cap S$ consists of all points $\lambda p + \mu q$ such that $(\lambda : \mu) \in \mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ is a root of the homogeneous polynomial $f_{pq}(\lambda, \mu) \overset{\text{def}}{=} f(\lambda p + \mu q) \in \Bbbk[\lambda, \mu]$. Over an algebraically closed field $\Bbbk$, this polynomial is either zero or a product of n non-zero homogeneous linear forms in λ, μ, possibly coinciding:

$$f(\lambda, \mu) = \prod_i(\alpha_i''\lambda - \alpha_i'\mu)^{s_i} = \prod_i \det{}^{s_i}\begin{pmatrix}\lambda & \alpha_i'\\ \mu & \alpha_i''\end{pmatrix}\,, \tag{4-34}$$

where $a_i = (\alpha_i' : \alpha_i'')$ are some mutually distinct points on $\mathbb{P}_1$ and $\sum_i s_i = n$. If $f_{pq} = 0$, then $\ell \subset S$. If $f_{pq} \neq 0$, then the intersection $\ell \cap S$ consists of the points $a_i = \alpha_i' p + \alpha_i'' q$. The exponent s_i of the linear form $\alpha_i''\mu - \alpha_i'\lambda$ in the factorization (4-34) is called the *intersection multiplicity* of the hypersurface S with the line ℓ at the point a_i, and is denoted by $(S, \ell)_{a_i}$. If $(S, \ell)_{a_i} = 1$, the intersection point a_i is called *simple* or *transversal*. Otherwise, the intersection of ℓ and S at a_i is called a *multiple*. The total number of intersections counted with their multiplicities equals the degree of S.

A line $\ell = (pq)$ passing through $p \in S$ is called *tangent* to S at p if either $\ell \subset S$ or $(S, \ell)_p \geqslant 2$. In other words, the line ℓ is tangent to S at p if the polynomial $f(p + tq) \in \Bbbk[t]$ either is the zero polynomial or has a multiple root at zero. The Taylor expansion[1] for $f(p + tq)$ at p starts with

$$f(p+tq) = t\binom{d}{1}\widetilde{f}(p^{n-1}, q) + t^2\binom{d}{2}\widetilde{f}(p^{n-2}, q^2) + \cdots\,.$$

[1]See 4-33 on p. 51.

Therefore the line $\ell = (pq)$ is tangent to S at p if and only if $\widetilde{f}(p^{n-1}, q) = 0$. This is the straightforward generalization of Proposition 2.2 on p. 17.

If $f(p^{n-1}, x)$ does not vanish identically as a linear form in x, the point p is called a *smooth point* of S. The hypersurface $S \subset \mathbb{P}(V)$ is called *smooth* if every point $p \in S$ is smooth. For a smooth $p \in S$ the linear equation $F(p^{n-1}, x) = 0$ on $x \in V$ defines a hyperplane in $\mathbb{P}(V)$ filled by the lines (pq) tangent to S at p. This hyperplane is called the *tangent space* to S at p and denoted by $T_p = \{x \in \mathbb{P}(V) \mid \widetilde{f}(p^{n-1}, x) = 0\}$.

If $f(p^{n-1}, x)$ is the zero linear form in x, the hypersurface S is called *singular* at p, and the point p is called a *singular point* of S. Since the coefficients of polynomial $\widetilde{f}(p^{n-1}, x) = \partial_x f(p)$, considered as a linear form in x, are equal to the partial derivatives of f evaluated at the point p by (4-32), the singularity of $p \in S = V(f)$ is expressed by the equations

$$\frac{\partial f}{\partial x_i}(p) = 0 \quad \text{for all } i\,,$$

in which case any line ℓ passing through p has $(S, \ell)_p \geqslant 2$, i.e., is tangent to S at p. Thus, the tangent lines to S at a singular point of S fill the whole ambient space $\mathbb{P}(V)$.

If q is either a smooth point on S or a point outside S, then the polar polynomial

$$\mathrm{pl}_q f(x) = \widetilde{f}(q, x^{n-1})$$

does not vanish identically as a homogeneous polynomial of degree $n-1$ in x, because otherwise, all partial derivatives of $\mathrm{pl}_q f(x) = \widetilde{f}(q, x^{n-1})$ in x would also vanish, and in particular,

$$\widetilde{f}(q^{n-1}, x) = \frac{\partial^{n-2}}{\partial q^{n-2}} \mathrm{pl}_q f(x) = 0$$

identically in x, meaning that q is a singular point of S, in contradiction with our choice of q. The zero set of the polar polynomial $\mathrm{pl}_q f \in S^{n-1}V^*$ is denoted by

$$\mathrm{pl}_q S \stackrel{\text{def}}{=} V(\mathrm{pl}_q f) = \{x \in \mathbb{P}(V) \mid \widetilde{f}(q, x^{n-1}) = 0\} \tag{4-35}$$

and called the *polar hypersurface* of the point q with respect to S. If S is a quadric, then $\mathrm{pl}_q S$ is exactly the polar hyperplane of q considered in n° 2.3.1 on p. 19. As in Corollary 2.2 on p. 17, for a hypersurface S of arbitrary degree, the intersection $S \cap \mathrm{pl}_q S$ coincides with the *apparent contour* of S viewed from the point q, that is, with the locus of all points $p \in S$ such that the line (pq) is tangent to S at p.

More generally, for an arbitrary point $q \in \mathbb{P}(V)$ the locus of points

$$\mathrm{pl}_q^{n-r} S \stackrel{\text{def}}{=} \left\{ x \in \mathbb{P}(V) \mid \widetilde{f}(q^{n-r}, x^r) = 0 \right\}$$

is called the *rth degree polar* of the point q with respect to S or *the rth degree polar* of S at q for $q \in S$. If the polynomial $\widetilde{f}(q^{n-r}, x^r)$ vanishes identically in x, we say that the rth degree polar is *degenerate*. Otherwise, the rth degree polar is a projective hypersurface of degree r. The linear[1] polar of S at a smooth point $q \in S$ is simply the tangent hyperplane to S at q: $\mathrm{pl}_q^{n-1} S = T_q S$. The quadratic polar $\mathrm{pl}_q^{n-2} S$ is the quadric passing through q and having the same tangent hyperplane at q as S. The cubic polar $\mathrm{pl}_q^{n-3} S$ is the cubic hypersurface passing through q and having the same quadratic polar at q as S, etc. The rth degree polar $\mathrm{pl}_q^{n-2} S$ at a smooth point $q \in S$ passes through q and has $\mathrm{pl}_q^{r-k} \mathrm{pl}_q^{n-r} S = \mathrm{pl}_q^{n-k} S$ for all $1 \leqslant k \leqslant r-1$, because

$$\mathrm{pl}_q^{r-k} \mathrm{pl}_q^{n-r} f(x) = \widetilde{\mathrm{pl}_q^{n-r} f}(q^{r-k}, x^k) = \widetilde{f}(q^{n-r}, q^{r-k}, x^k) = \widetilde{f}(q^{n-k}, x^k) = \mathrm{pl}_q^{n-k} f(x)\,.$$

[1] That is, of the first degree.

4.5.4 Linear support of a homogeneous polynomial. For a polynomial $f \in S^n V^*$, we write $\operatorname{Supp} f$ for the minimal[1] vector subspace $W \subset V^*$ such that $f \in S^n W$, and call it the *linear support* of f. Over a field of zero characteristic, $\operatorname{Supp} f = \operatorname{Supp} \widetilde{f}$, where $\widetilde{f} \in \operatorname{Sym}^n V^* \subset V^{*\otimes n}$ is the complete polarization of f. By Theorem 4.1, $\operatorname{Supp} \widetilde{f}$ is linearly generated by the images of the $(n-1)$-tuple contraction maps

$$c_t^J : V^{\otimes(n-1)} \to V^*, \quad t \mapsto c^{1,\,2,\,\dots\,,\,(n-1)}_{j_1,j_2,\dots,j_{n-1}}(t \otimes \widetilde{f}),$$

coupling all the $(n-1)$ factors of $V^{\otimes(n-1)}$ with some $n-1$ factors of $\tilde{t} \in V^{*\otimes n}$ in order indicated by the sequence $J = (j_1, j_2, \dots, j_{n-1})$. For the symmetric tensor $\widetilde{f}$, such a contraction does not depend on J and maps every decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1}$ to the linear form on V proportional to the derivative $\partial_{v_1}\partial_{v_2} \dots \partial_{v_{n-1}} f \in V^*$. Thus, $\operatorname{Supp}(f)$ is linearly generated by all $(n-1)$-tuple partial derivatives

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}} \frac{\partial^{m_2}}{\partial x_2^{m_2}} \cdots \frac{\partial^{m_d}}{\partial x_d^{m_d}} f(x), \quad \text{where} \sum m_\nu = n-1. \tag{4-36}$$

The coefficient of x_i in the linear form (4-36) depends only on the coefficients of monomial

$$x_1^{m_1} \dots x_{i-1}^{m_{i-1}} x_i^{m_i+1} x_{i+1}^{m_{i+1}} \dots x_d^{m_d}$$

in f. If we write the polynomial f as

$$f = \sum_{\nu_1+\dots+\nu_d=n} \frac{n!}{\nu_1!\,\nu_2!\,\cdots\,\nu_d!}\, a_{\nu_1\nu_2\dots\nu_d} x_1^{\nu_1} x_2^{\nu_2} \cdots x_d^{\nu_d}, \tag{4-37}$$

the linear form (4-36) turns to

$$n! \cdot \sum_{i=1}^{d} a_{m_1\dots m_{i-1}(m_i+1)m_{i+1}\dots m_d} x_i. \tag{4-38}$$

Totally, we get $\binom{n+d-2}{d-1}$ such the linear forms staying in bijection with the nonnegative integer solutions $m_1, m_2, \dots, m_d$ of the equation $m_1 + m_2 + \dots + m_d = n-1$.

PROPOSITION 4.3
Let $\Bbbk$ be a field of zero characteristic, V a finite dimensional vector space over $\Bbbk$, and $f \in S^n V^*$ a polynomial written in the form (4-37) in some basis of V^*. If $f = \varphi^n$ for some linear form $\varphi \in V^*$, then the $d \times \binom{n+d-2}{d-1}$ matrix built from the coefficients of linear forms (4-38) has rank 1. In this case, there are at most n linear forms $\varphi \in V^*$ such that $\varphi^n = f$, and they differ from one another by multiplications by the nth roots of unity laying in $\Bbbk$. For algebraically closed field $\Bbbk$, the converse is also true: if all the linear forms (4-38) are proportional, then $f = \varphi^n$ for some linear form φ proportional to the forms (4-38).

PROOF. The equality $f = \varphi^n$ means that $\operatorname{Supp}(f) \subset V^*$ is the 1-dimensional subspace spanned by φ. In this case, all linear forms (4-38) are proportional to φ. Such a form $\psi = \lambda\varphi$ has $\psi^n = f$ if and only if $\lambda^n = 1$ in $\Bbbk$. Conversely, let all the linear forms (4-38) be proportional, and $\psi \neq 0$ be one of them. Then, $\operatorname{Supp}(f) = \Bbbk \cdot \psi$ is the 1-dimensional subspace spanned by ψ. Hence, $f = \lambda\psi^n$ for some $\lambda \in \Bbbk$, and therefore, $f = \varphi^n$ for[2] $\varphi = \sqrt[n]{\lambda} \cdot \psi$. □

[1] With respect to inclusions.
[2] Here we use that $\Bbbk$ is algebraically closed.

4.5.5 The Veronese varieties $V(n,k)$. The *Veronese map*

$$v_{k,n}: \mathbb{P}(V^*) \hookrightarrow \mathbb{P}(S^nV^*), \quad \psi \mapsto \psi^n, \tag{4-39}$$

for $\dim V = k+1$ embeds $\mathbb{P}_k$ into $\mathbb{P}_N$, where $N = \binom{n+k}{k} - 1$. The image of map (4-39) is called the *Veronese variety* and denoted by $V(k,n) \subset \mathbb{P}(S^nV^*)$. It consists of perfect n th powers φ^n of linear forms $\varphi \in V^*$ considered up to proportionality. It follows from Proposition 4.3 that $V(n,k)$ is indeed an algebraic projective variety described by a system of quadratic equations asserting the vanishing of all 2×2-minors in $d \times \binom{n+d-2}{d-1}$ matrix formed by the coefficients of the linear forms (4-38). For example, a homogeneous polynomial in two variables $f(x_0,x_1) = \sum_{k=0}^n a_k\binom{n}{k}x_0^{n-k}x_1^k$ has

$$\frac{\partial^{n-1}f}{\partial x_0^{n-i-1}\partial x_1^i} = n!\cdot(a_ix_0 + a_{i+1}x_1).$$

Hence, the image of the Veronese embedding $v_{1,n}: \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ is described by the condition

$$\operatorname{rk}\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1,$$

which agrees with Example 1.4 on p. 11 and is equivalent to a system of quadratic equations

$$\det\begin{pmatrix} a_i & a_j \\ a_{i+1} & a_{j+1} \end{pmatrix} = 0$$

on the coefficients a_i of the polynomial f. A polynomial f satisfies these equations if and only if $f = \varphi^n$ for some linear form $\varphi = \alpha_0x_0 + \alpha_1x_1$, and in this case $(\alpha_0 : \alpha_1) = (a_i : a_{i+1})$ for all i.

4.6 Polarization of grassmannian polynomials. The quotient map $V^{\otimes n} \twoheadrightarrow \Lambda^nV$ sends every summand of the basis alternating tensor (4-26)

$$e_{\langle i_1,i_2,\dots,i_n\rangle} \overset{\text{def}}{=} \sum_{g\in S_n} \operatorname{sgn}(g)\cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}}$$

to the same grassmannian monomial $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. Thus, this map sends $e_{\langle i_1,i_2,\dots,i_n\rangle}$ to $n!\,e_I$, and therefore, over a field of zero characteristic, the factorization through the alternating relations assigns the isomorphism $\operatorname{Alt}^n V \xrightarrow{\sim} \Lambda^nV$. By analogy with the usual commutative polynomials, the inverse isomorphism is denoted by $\operatorname{pl}: \Lambda^nV \xrightarrow{\sim} \operatorname{Alt}^n V$, $\omega \mapsto \widetilde{\omega}$, and called the *complete polarization* of grassmannian polynomials.

4.6.1 Duality. For a finite dimensional vector space V over a field of zero characteristic, there is the perfect pairing between the spaces Λ^nV and Λ^nV^* coupling $\tau \in \Lambda^nV$ and $\omega \in \Lambda^nV^*$ to the complete contraction of their complete polarizations $\tilde{\tau} \in V^{\otimes n}$ and $\widetilde{\omega} \in V^{*\otimes n}$.

EXERCISE 4.21. Convince yourself that the non zero couplings between the basis monomials $e_I \in \Lambda^nV$ and $x_J \in \Lambda^nV^*$ are exhausted by $\langle\, e_I\,,\, x_I\,\rangle = 1/n!$.

4.6.2 Partial derivatives in the exterior algebra. Given a covector $\psi \in V^*$, we write

$$\operatorname{pl}_\psi: \Lambda^nV \to \Lambda^{n-1}V$$

for the composition of inner multiplication $i_\psi: V^{\otimes n} \to V^{\otimes(n-1)}$ by ψ with preceding complete polarization $\operatorname{pl}: \Lambda^nV \xrightarrow{\sim} \operatorname{Alt}^n V$ and subsequent factorization $\alpha: V^{\otimes(n-1)} \twoheadrightarrow \Lambda^{n-1}V$ through the alternating relations[1]. Thus, pl_v fits in the commutative diagram

$$\begin{array}{ccc} V^{*\otimes n} \supset \mathrm{Skew}^n V^* & \xrightarrow{\ i_\psi\ } & V^{*\otimes(n-1)} \\ \mathrm{pl}\uparrow\wr & & \downarrow\alpha \\ \Lambda^n V^* & \xrightarrow{\ \mathrm{pl}_\psi\ } & \Lambda^{n-1}V^* \end{array} \tag{4-40}$$

similar to the diagram from formula (4-31) on p. 50. By analogy with n° 4.5.2, the polynomial

$$\partial_\psi \omega \overset{\text{def}}{=} \deg\omega \cdot \mathrm{pl}_\psi \omega$$

is called the *derivative* of homogeneous grassmannian polynomial $\omega \in \Lambda^n V$ in direction of covector $\psi \in V^*$. Since $\mathrm{pl}_\psi\omega$ is linear in ψ, the derivation along $\psi = \sum \alpha_i x_i$ splits as $\partial_\psi = \sum \alpha_i \partial_{x_i}$. If ω does not depend on e_i, then $\partial_{x_i}\omega = 0$. Therefore, a nonzero contribution to $\partial_\psi e_I$ is given only by the derivations ∂_{x_i} for $i \in I$.

EXERCISE 4.22. Check that $\partial_{x_{i_1}} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} = e_{i_2} \wedge e_{i_3} \wedge \ldots \wedge e_{i_n}$ for every collection of indexes $i_1, i_2, \ldots, i_n$, not necessary increasing.

It follows from Exercise 4.22 that

$$\begin{aligned} \partial_{x_{i_k}} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} &= \partial_{x_{i_k}} (-1)^{k-1} e_{i_k} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \ldots e_{i_n} \\ &= (-1)^{k-1} \partial_{x_{i_k}} e_{i_k} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \ldots e_{i_n} \\ &= (-1)^{k-1} e_{i_1} \wedge \ldots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \ldots e_{i_n} . \end{aligned}$$

In other words, the derivation of a monomial along the basis covector dual to the kth variable from the left in the monomial behaves as $(-1)^{k-1}\partial/\partial e_{i_k}$, where the *grassmannian partial derivative* $\partial/\partial e_i$ takes e_i to 1 and annihilates all e_j with $j \neq i$, exactly as in the symmetric case. However, the sign $(-1)^k$ in the previous formula forces the grassmannian partial derivatives to satisfy *the grassmannian Leibniz rule*, which differs from the usual one by an extra sign.

EXERCISE 4.23 (THE GRASSMANNIAN LEIBNIZ RULE). For any homogeneous grassmannian polynomials $\omega, \tau \in \Lambda V$ and a covector $\psi \in V$, prove that

$$\partial_\psi(\omega \wedge \tau) = \partial_\psi(\omega) \wedge \tau + (-1)^{\deg\omega} \omega \wedge \partial_\psi(\tau) . \tag{4-41}$$

Since the grassmannian polynomials are linear in each variable, $\partial_\psi^2 \omega = 0$ for all $\psi \in V$, $\omega \in \Lambda V$. The relation $\partial_\psi^2 = 0$ forces the grassmannian derivatives to be super-commutative, that is,

$$\forall\, \psi, \xi \in V^* \quad \partial_\psi \partial_\xi = -\partial_\xi \partial_\psi .$$

4.6.3 Linear support of a homogeneous grassmannian polynomial. The *linear support* $\mathrm{Supp}\,\omega$ of a homogeneous grassmannian polynomial ω of degree n is defined to be the minimal[2] vector subspace $W \subset V$ such that $\omega \in \Lambda^n W$. It coincides with the linear support of the complete polarization $\widetilde{\omega} \in \mathrm{Skew}^n V$, and is linearly generated by all $(n-1)$-tuple partial derivatives[3]

$$\partial_J \omega \overset{\text{def}}{=} \partial_{x_{j_1}} \partial_{x_{j_2}} \ldots \partial_{x_{j_{n-1}}} \omega = \frac{\partial}{\partial e_{j_1}} \frac{\partial}{\partial e_{j_2}} \cdots \frac{\partial}{\partial e_{j_{n-1}}} \omega ,$$

[1] Which is the linear map corresponding to the alternating multiplication of covectors from formula (4-17) on p. 43 by the universal property of tensor product.

[2] With respect to inclusions.

[3] Compare with n° 4.5.4 on p. 53.

where $J = j_1 j_2 \dots j_{n-1}$ runs through all sequences of $n-1$ different indexes taken from the set $\{1, 2, \dots, d\}$, $d = \dim V$. Up to a sign, the order of indexes in J is not essential, and we will not assume the indexes to be increasing, because this simplifies the notations in what follows.

Let us expand ω as a sum of basis monomials

$$\omega = \sum_I a_I e_I = \sum_{i_1 i_2 \dots i_n} \alpha_{i_1 i_2 \dots i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}, \tag{4-42}$$

where $I = i_1 i_2 \dots i_n$ also runs through the n-tuples of different but non necessary increasing indexes, and the coefficients $\alpha_{i_1 i_2 \dots i_n} \in \Bbbk$ are alternating in $i_1 i_2 \dots i_n$. Nonzero contributions to $\partial_J \omega$ are given only by the monomials $a_I e_I$ with $I \supset J$. Therefore, up to a common sign,

$$\partial_J \omega = \pm \sum_{i \notin J} \alpha_{j_1 j_2 \dots j_{n-1} i} e_i. \tag{4-43}$$

PROPOSITION 4.4
The following conditions on a grassmannian polynomial $\omega \in \Lambda^n V$ written in the form (4-42) are equivalent:

1) $\omega = u_1 \wedge u_2 \wedge \cdots \wedge u_n$ for some $u_1, u_2, \dots, u_n \in V$
2) $u \wedge \omega = 0$ for all $u \in \operatorname{Supp}(\omega)$
3) for any two collections $i_1 i_2 \dots i_{m+1}$ and $j_1 j_2 \dots j_{m-1}$ consisting of $n+1$ and $n-1$ different indexes, the following *Plücker relation* holds

$$\sum_{\nu=1}^{m+1} (-1)^{\nu-1} a_{j_1 \dots j_{m-1} i_\nu} a_{i_1 \dots \hat{i}_\nu \dots i_{m+1}} = 0, \tag{4-44}$$

where the hat in $a_{i_1 \dots \hat{i}_\nu \dots i_{m+1}}$ means that the index i_ν should be removed.

PROOF. Condition (1) holds if and only if ω belongs to the top homogeneous component of its linear span, $\omega \in \Lambda^{\dim \operatorname{Supp}(\omega)} \operatorname{Supp}(\omega)$. Condition (2) means the same because of the following exercise.

EXERCISE 4.24. Show that $\omega \in \Lambda U$ is homogeneous of degree $\dim U$ if and only if $u \wedge \omega = 0$ for $u \in U$.

The Plücker relation (4-44) asserts the vanishing of the coefficient of $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{m+1}}$ in the product $\left(\partial_{j_1 \dots j_{m-1}} \omega\right) \wedge \omega$. In other words, (4-44) is the coordinate form of condition (2) written for vector $u = \partial_{j_1 \dots j_{m-1}} \omega$ from the formula (4-43). Since these vectors linearly generate the subspace $\operatorname{Supp}(\omega)$, the whole set of the Plücker relations is equivalent to the condition (2). □

EXAMPLE 4.8 (THE PLÜCKER QUADRIC)
Let $n = 2$, $\dim V = 4$, and e_1, e_2, e_3, e_4 be a basis of V. Then the expansion (4-42) for $\omega \in \Lambda^2 V$ looks like $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$, where the coefficients a_{ij} form the alternating 4×4 matrix. The Plücker relation corresponding to $(i_1, i_2, i_3) = (2, 3, 4)$ and $j_1 = 1$ is

$$a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0. \tag{4-45}$$

All other choices of (i_1, i_2, i_3) and $j_1 \notin \{i_1, i_2, i_3\}$ lead to exactly the same relation.

EXERCISE 4.25. Check this.

For $j_1 \in \{i_1, i_2, i_3\}$ we get the trivial equality $0 = 0$. Thus, for $\dim V = 4$, the set of decomposable grassmannian quadratic forms $\omega \in \Lambda^2 V$ is described by just one quadratic equation (5-2).

EXERCISE 4.26. Convince yourself that the equation (5-2) on $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$ is equivalent to the condition $\omega \wedge \omega = 0$.

4.6.4 The Grassmannian varieties and Plücker embeddins. For a vector space V of dimension d, the set of all vector subspaces $U \subset V$ of dimension m is denoted by $\mathrm{Gr}(m, V)$ and called the *grassmannian*. When the origin of V is not essential or $V = \Bbbk^d$, we write $\mathrm{Gr}(m, d)$ instead of $\mathrm{Gr}(m, V)$. Thus, $\mathrm{Gr}(1, V) = \mathbb{P}(V)$, $\mathrm{Gr}(\dim V - 1, V) = \mathbb{P}(V^*)$. The grassmannian $\mathrm{Gr}(m, V)$ is embedded into the projective space $\mathbb{PP}(\Lambda^m V)$ by means of the *Plücker map*

$$p_m : \mathrm{Gr}(m, V) \to \mathbb{P}(\Lambda^m V), \quad U \mapsto \Lambda^m U \subset \Lambda^m V \tag{4-46}$$

sending every subspace $U \subset V$ of dimension m to its highest exterior power $\Lambda^m U$, which is a subspace of dimension 1 in $\Lambda^m V$. If U is spanned by vectors $u_1, u_2, \dots, u_m$, then up to proportionality, $p_m(U) = u_1 \wedge u_2 \wedge \cdots \wedge u_m$.

EXERCISE 4.27. Check that the Plücker map is injective.

The image of map (4-46) consists of all grassmannian polynomials $\omega \in \Lambda^m V$ completely factorisable into a product of m vectors. Such polynomials are called *decomposable*. By Proposition 4.4 they form a projective algebraic variety described by the system of quadratic equations (4-44) on the coefficients of expansion (4-42).

REMARK 4.1. From the algebraic viewpoint, the grassmannian variety $\mathrm{Gr}(k, m) \subset \mathbb{P}(\Lambda^m V)$ is a super-commutative version of the Veronese variety $V(k, m) \subset \mathbb{P}(S^m V)$. Both consist of most degenerated non-zero homogeneous polynomials of degree m in the sense that the linear support of polynomial has the minimal possible dimension which equals 1 for a commutative polynomial, and equals m for a grassmannian polynomial of degree m.

EXAMPLE 4.9 (THE GRASSMANNIANS $\mathrm{Gr}(2, V)$)
The Plücker embedding identifies the grassmannian $\mathrm{Gr}(2, V)$ with the set of decomposable grassmannian quadratic forms $\omega \in \Lambda^2 V$, that is, $\omega = u \wedge w$ for some $u, w \in V$. Note that every such ω has $\omega \wedge \omega = u \wedge w \wedge u \wedge w = 0$. For an arbitrary $\omega \in \Lambda^2 V$, there exists a basis $\xi_1, \xi_2, \dots, \xi_d$ in V such that[1] $\omega = \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots$. If this sum contains more than one term, then the monomial $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$ appears in $\omega \wedge \omega$ with the coefficient 2 and therefore, $\omega \wedge \omega \neq 0$. Thus, such ω is not decomposable. We conclude that $\omega \in \Lambda^2 V$ is decomposable if and only if $\omega \wedge \omega = 0$.

For $\dim V = 4$, the squares of forms $\omega \in \Lambda^2 V$ lie in the space $\Lambda^4 V$ of dimension 1. In this case, the condition $\omega \wedge \omega = 0$ for $\omega = \sum_{i,j} a_{ij} e_i \wedge e_j$ is expressed by just one quadratic equation

$$a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0, \tag{4-47}$$

which agrees with the equation (5-2) from Example 4.8 on p. 56. We conclude that the Plücker embedding identifies the grassmannian $\mathrm{Gr}(2, 4) = \mathrm{Gr}(2, V)$ with the quadric (4-47) in $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. This quadric is called the *Plücker quadric*.

EXAMPLE 4.10 (THE SEGRE VARIETIES REVISITED[2])
Let $W = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ be a direct sum of finite dimensional vector spaces V_i. For every collection of non-negative integers $m_1, m_2, \dots, m_n$ such that $m_i \leqslant \dim V_i$, put $k = \sum_\nu m_\nu$ and

[1] See Example 4.5 on p. 47.
[2] See n° 4.1.2 on p. 39.

denote by $W_{m_1,m_2,\dots,m_n} \subset \Lambda^k W$ the linear span of all products $w_1 \wedge w_2 \wedge \cdots \wedge w_k$ formed by m_1 vectors taken from V_1, m_2 vectors taken from V_2, etc.

EXERCISE 4.28. Show that the well defined isomorphism of vector spaces

$$\Lambda^{m_1}V_1 \otimes \Lambda^{m_2}V_2 \otimes \cdots \otimes \Lambda^{m_n}V_n \xrightarrow{\sim} W_{m_1,m_2,\dots,m_n}$$

is assigned by prescription $\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n \mapsto \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$, and verify that

$$\Lambda^k W = \bigoplus_{m_1,m_2,\dots,m_n} W_{m_1,m_2,\dots,m_n} \simeq \bigoplus_{m_1,m_2,\dots,m_n} \Lambda^{m_1}V_1 \otimes \Lambda^{m_2}V_2 \otimes \cdots \otimes \Lambda^{m_n}V_n .$$

We conclude that the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ can be identified with the component $W_{1,1,\dots,1} \subset \Lambda^n W$. Under this identification, the decomposable tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ go to the decomposable grassmannian monomials $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. Therefore, the Segre variety from n° 4.1.2 on p. 39 is the intersection of the grassmannian variety $\mathrm{Gr}(n, W) \subset \mathbb{P}(\Lambda^n W)$ with the projective subspace $\mathbb{P}\left(W_{1,1,\dots,1}\right) \subset \mathbb{P}(\Lambda^n W)$. In particular, the Segre variety is indeed an algebraic variety described by the system of quadratic equations from Proposition 4.4 on p. 56 restricted onto the linear subspace $W_{1,1,\dots,1} \subset \Lambda^n W$.

§5 Grassmannian varieties in more details

5.1 The Plücker quadric and grassmannian Gr(2,4). Let us fix a vector space V of dimension 4. The grassmannian $\mathrm{Gr}(2,V) = \mathrm{Gr}(2,4)$ parameterizes the vector subspaces $U \subset V$ of dimension 2, or equivalently, the lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$. The *Plücker embedding*

$$\mathfrak{u} : \mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V), \quad U \mapsto \Lambda^2 U, \quad (ab) \mapsto a \wedge b \tag{5-1}$$

sends every 2-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$, or equivalently, every line $(ab) \subset \mathbb{P}(V)$ to the point $a \wedge b \in \mathbb{P}(\Lambda^2 V)$. It assigns the bijection between the grassmannian $\mathrm{Gr}(2,4)$ and the *Plücker quadric*

$$P \stackrel{\text{def}}{=} \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\} \tag{5-2}$$

which consists of all decomposable grassmannian quadratic forms $\omega = a \wedge b$, $a, b \in V$, see Example 4.9 on p. 57.

Let us fix a basis e_0, e_1, e_2, e_3 in V, the monomial basis $e_{ij} \stackrel{\text{def}}{=} e_i \wedge e_j$ in $\Lambda^2 V$, and write x_{ij} for the homogeneous coordinates in $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ with respect to the latter basis. The computation

$$\Big(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) \wedge \Big(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) = 2\left(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}\right) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$$

shows that P is described by the non-degenerated quadratic equation $x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12}$.

Exercise 5.1. Check that the Plücker embedding (5-1) takes the subspace spanned by vectors $a = \sum \alpha_i e_i$, $b = \sum \beta_j e_j$ to the point with coordinates $x_{ij} = \alpha_i\beta_j - \alpha_j\beta_i$, that is, sends a matrix $\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$ to the collection of its six 2×2-minors $x_{ij} = \det \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}$.

In coordinate-free terms, the Plücker quadric is described as follows. There exists a unique up to proportionality bilinear form $\widetilde{q}$ on $\Lambda^2 V$ defined by prescription

$$\forall\, \omega_1, \omega_2 \in \Lambda^2 V \quad \omega_1 \wedge \omega_2 = \widetilde{q}(\omega_1, \omega_2) \cdot \delta\,, \tag{5-3}$$

where $\delta \in \Lambda^4 V \simeq \Bbbk$ is an arbitrary non zero vector[1]. This form is symmetric, because $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$ for even grassmannian polynomials. Obviously, $P = V(q)$ for the quadratic form $q(\omega) = \widetilde{q}(\omega, \omega)$ corresponding to $\widetilde{q}$.

Lemma 5.1
Two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ are intersecting if and only if $\widetilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = 0$ in $\mathbb{P}_5$.

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and we can choose a basis $e_0, e_1, e_2, e_3 \in V$ such that $\ell_1 = (e_0e_1)$, $\ell_2 = (e_2e_3)$. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $\ell_1 = (ab)$, $\ell_2 = (ac)$ are intersecting in a, then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = a \wedge b \wedge a \wedge c = 0$. □

Remark 5.1. The injectivity of (5-1) becomes obvious[2] after Lemma 5.1. Indeed, for any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersects ℓ_1 and does not intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$. This forces $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$.

[1] Since $\dim \Lambda^4 V = 1$, such a vector is unique up to proportionality.
[2] Compare with Exercise 4.27 on p. 57.

COROLLARY 5.1
For every point $p = \mathfrak{u}(\ell) \in P$, the intersection $P \cap T_p P$ in $\mathbb{P}_5$ consists of all points $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$ in $\mathbb{P}_3$.

PROOF. This follows from Lemma 5.1 and Proposition 2.2 on p. 17. □

5.1.1 Nets and pencils of lines in $\mathbb{P}_3$. A family of lines on $\mathbb{P}_3$ is called a *net* if the Plücker embedding sends it to a plane $\pi \subset P \subset \mathbb{P}_5$. Every plane $\pi \subset P$ is spanned by a triple of non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, and lies in the intersection of tangent spaces to P at these points: $\pi \subset P \cap T_{p_1} P \cap T_{p_2} P \cap T_{p_3} P$. It follows from the Lemma 5.1 and Corollary 5.1 that the corresponding net of lines in $\mathbb{P}_3$ consists of all lines intersecting three given pairwise intersecting lines ℓ_1, ℓ_2, ℓ_3. Since three mutually intersecting lines have to be either concurrent or coplanar, there are exactly two different types of line nets in $\mathbb{P}_3$:

α-net consists of lines passing through a given point $a \in \mathbb{P}_3$ and corresponds to *α-plane* $\pi_\alpha(a) \subset P$ spanned by Plücker's images of three non-coplanar lines passing through a

β-net consists of lines laying in a given plane $\Pi \in \mathbb{P}_3$ and corresponds to *β-plane* $\pi_\beta(\Pi) \subset P$ spanned by Plücker's images of three non-concurrent lines laying in Π.

Any two planes of the same type have exactly one intersection point:

$$\pi_\beta(\Pi_1) \cap \pi_\beta(\Pi_2) = \mathfrak{u}(\Pi_1 \cap \Pi_2)\,, \quad \pi_\alpha(a_1) \cap \pi_\alpha(a_2) = \mathfrak{u}((a_1 O_2))\,.$$

Two planes of different types $\pi_\beta(\Pi)$, $\pi_\alpha(a)$ are either not intersecting (if $a \notin \Pi$) or intersecting along a line (if $a \in \Pi$). In the latter case the intersection line depicts the pencil of lines in $\mathbb{P}_3$ passing through a and laying in Π.

EXERCISE 5.2. Show that there are no other pencils of lines in $\mathbb{P}_3$, i.e., every line laying on $P \subset \mathbb{P}_5$ has the form $\pi_\beta(\Pi) \cap \pi_\alpha(a)$ for some $a \in \Pi \subset \mathbb{P}_3$.

EXERCISE 5.3. Convince yourself that the assignment $U \mapsto \operatorname{Ann} U$ establishes the bijection $\operatorname{Gr}(2, V) \xrightarrow{\sim} \operatorname{Gr}(2, V^*)$ sending α-planes to β-planes and vice versa.

5.1.2 Cell decomposition of P. Let us fix a point $p \in P$ and a hyperplane $H \simeq \mathbb{P}_3$ laying inside $T_p P \simeq \mathbb{P}_4$ and complementary to p within this $\mathbb{P}_4$. The intersection $C = P \cap T_p P$ is the simple cone with vertex p over a smooth quadric $G = H \cap P$, which can be thought of as the Segre quadric in $\mathbb{P}_3 = H$. Fix a point $p' \in G$ and write π_α, π_β for the planes spanned by p and two lines laying on G and passing through p'. Associated with these data is the following stratification of the Plücker quadric P by closed subvarieties shown on fig. 5◇1 on p. 61:

$$\begin{array}{ccccccc} & & & \pi_\alpha & & & \\ & & \nearrow & & \searrow & & \\ p & \hookrightarrow & \pi_\alpha \cap \pi_\beta & & & C & \hookrightarrow P \\ & & \searrow & & \nearrow & & \\ & & & \pi_\beta & & & \end{array} \tag{5-4}$$

For every stratum σ of this stratification, the complement to the union of all strata contained in σ is naturally identified with an affine space. This leads to the following decomposition of Gr(2, 4) in disjoint union of affine spaces:

$$\mathrm{Gr}(2,4) \;=\; \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4 \,.$$

The leftmost $\mathbb{A}^0$ is the point p. Then goes $\mathbb{A}^1$, which is the complement to p within the projective line $(pp') = \pi_\alpha \cap \pi_\beta$. Then go two affine planes $\mathbb{A}^2$, the complements to (pp') within the projective planes π_α and π_β respectively. Then goes $\mathbb{A}^3$, which is the complement to $\pi_\alpha \cup \pi_\beta$ within the cone $C = P \cap T_p P$, which is the linear join of G and p. This complement is isomorphic to the direct product of $\mathbb{A}^1$, which is the cone generator punctured at the vertex of cone, and $\mathbb{A}^2 = G \smallsetminus T_{p'} G$. The rightmost piece $\mathbb{A}^4 = P \smallsetminus C$. The identifications $G \smallsetminus T_{p'} G = \mathbb{A}^2$ and $P \smallsetminus T_{p'} P = \mathbb{A}^4$ made on the last two steps are based on the Lemma 5.2 following below.

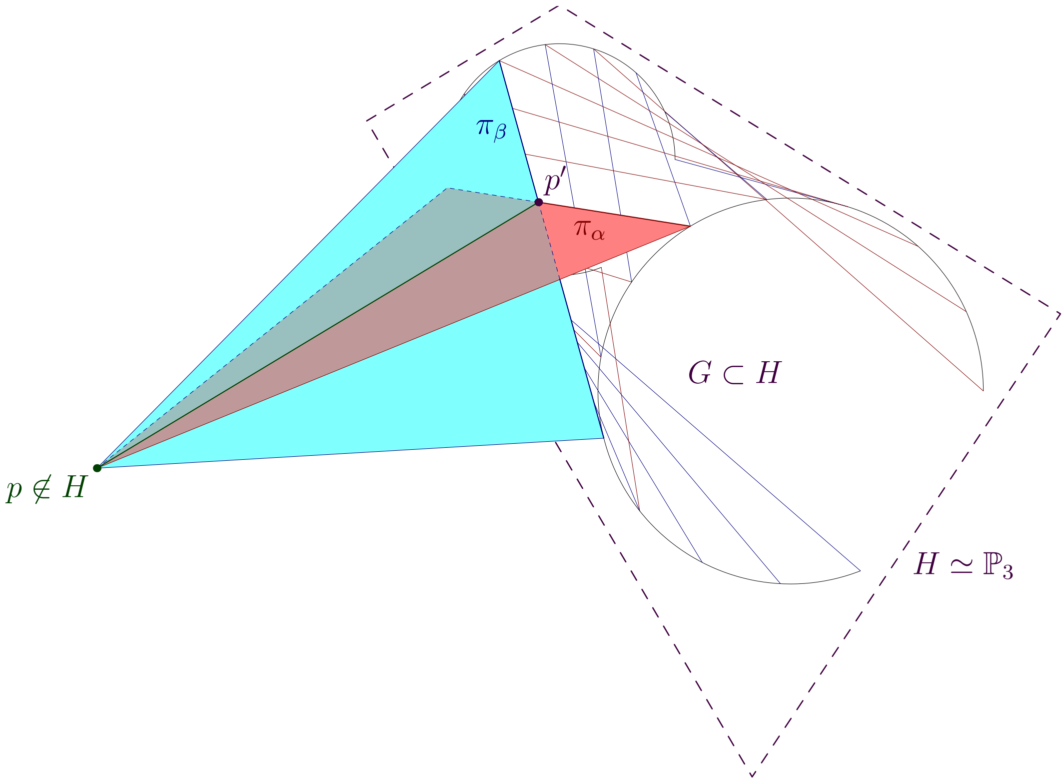


Fig. 5◇1. The cone $C = P \cap T_p P$ viewed within $\mathbb{P}_4 = T_p P$.

Lemma 5.2

For every smooth quadric $Q \subset \mathbb{P}_n$, point $p \in Q$, and hyperplane $\Pi \not\ni p$, the projection $p : Q \to \Pi$ from p to Π establishes a bijection between $Q \smallsetminus T_p Q$ and $\mathbb{A}^{n-1} = \Pi \smallsetminus T_p Q$.

Proof. Every non-tangent line passing through p intersects Q in exactly one point other than p. All these lines stay in bijection with the points of $\Pi \smallsetminus T_p Q \simeq \mathbb{A}^{n-1}$. □

Exercise 5.4. If you have some experience in CW-topology, show that the integer homology groups of complex grassmannian Gr(2, 4) are

$$H_m\left(\mathrm{Gr}(2,\mathbb{C}^4),\mathbb{Z}\right)=\begin{cases}0 & \text{for odd } m\leqslant 7 \text{ and all } m>8\\ \mathbb{Z} & \text{for } m=0,\ 2,\ 6,\ 8\\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } m=4\end{cases}$$

Try to compute the integer homologies $H_m\left(\mathrm{Gr}(2,\mathbb{R}^4),\mathbb{Z}\right)$ of the real grassmannian Gr(2, 4).

5.1.3 Lagrangian grassmannian LGr(2,4) and lines on a smooth quadric in $\mathbb{P}_4$. Let a vector space V of dimension 4 be equipped with a non-degenerated alternating bilinear form Ω. A line $\ell=(ab)\subset\mathbb{P}(V)$ is called *lagrangian* if $\Omega(u,w)=0$ for all $u,w\in\ell$, or equivalently, if $\Omega(a,b)=0$. The set of all lagrangian lines is called the *lagrangian grassmannian* and denoted by $\mathrm{LGr}(2,4)\subset\mathrm{Gr}(2,4)$. Let us show that the Plücker embedding sends LGr(2, 4) to a smooth hyperplane section of the Plücker quadric, that is, to a smooth quadric in $\mathbb{P}_4$.

Associated with Ω is the linear form $\Omega':\ \Lambda^2V\to\Bbbk$, $a\wedge b\mapsto\Omega(a,b)$. Let us fix a non-zero vector $\delta\in\Lambda^4V$. Since the bilinear form $\widetilde{q}$ on Λ^2V defined in formula (5-3) on p. 59 is non-degenerate, its correlation map $\widehat{q}:\ \Lambda^2V\simeq\left(\Lambda^2V\right)^*$ is an isomorphism. Hence, there exists a unique grassmannian quadratic form $\omega=\widehat{q}^{-1}(\Omega')\in\Lambda^2V$ such that

$$\forall\, a,b\in V\quad \omega\wedge a\wedge b=\Omega(a,b)\cdot\delta\,. \tag{5-5}$$

Write $W=\mathrm{Ann}\,\Omega'\subset\Lambda^2V$ for the orthogonal complement to ω with respect to the Plücker quadratic form q. The projectivization $Z=\mathbb{P}(W)\simeq\mathbb{P}_4\subset\mathbb{P}_5$ is the polar hyperplane of ω with respect to the Plücker quadric $P\subset\mathbb{P}(\Lambda^2V)$.

Exercise 5.5. Verify that $\omega\notin P$.

Hence, the intersection $R=Z\cap P$ is a smooth quadric within $\mathbb{P}_4=Z$. The points of this quadric stay in bijection with the lagrangian lines in $\mathbb{P}(V)$, because the formulas (5-5), (5-3) say together that a line $(ab)\subset\mathbb{P}_3$ is lagrangian if and only if $\widetilde{q}(\omega,\ a\wedge b)=0$. Thus, $\mathrm{LGr}(2,4)=R$ is a smooth quadric in $\mathbb{P}_4=Z$.

It follows from the general theory developed in n° 2.6 on p. 23 that R does not contain planes but every point $r\in R$ is the vertex of cone $R\cap T_rR$, the linear join of r with a smooth conic in a plane complementary to p within $T_pR\simeq\mathbb{P}_3$.

Definition 5.1 (the Fano variety of a projective variety)
The set of lines laying on a projective algebraic variety X is called the *Fano variety* of X and denoted by $F(X)$.

Proposition 5.1
For every point $p\in\mathbb{P}(V)$, the lagrangian lines $\ell\subset\mathbb{P}(V)$ passing through p form a pencil. Sending p to this pencil assigns the bijection $\mathbb{P}(V)\simeq F\big(\mathrm{LGr}(2,V)\big)$.

Proof. Every pencil of lines in $\mathbb{P}_3=\mathbb{P}(V)$ is mapped by the Plücker embedding to a line $L\subset P$, which has the form[1] $L=\pi_p\cap\pi(\Pi)$ for some point p and plane Π in $\mathbb{P}_3$ such that $p\in\Pi$. In other words, L consists of all lines passing through p and laying in Π. For $L\subset R=P\cap Z$ all these lines

[1] See n° 5.1.1, especially Exercise 5.2 on p. 60.

are lagrangian. On the other hand, a line $(px) \subset \mathbb{P}(V)$ is lagrangian if and only if $\Omega(p, x) = 0$. Hence, every lagrangian line passing through p lies in the orthogonal plane to p with respect to the form Ω and therefore, belongs to the pencil L. This proves the first statement. The second is obvious from the discussion preceding the proposition. □

5.2 The homogeneous, Plücker's, and affine coordinates on Gr(k,m). The general grassmannian $\mathrm{Gr}(k, m)$, which parameterizes the vector subspaces of dimension k in $V = \Bbbk^m$, is a straightforward generalization of the projective space $\mathbb{P}_{m-1} = \mathrm{Gr}(1, m)$ attached to V. If a basis $e_1, e_2, \dots, e_d$ in V is fixed, then a vector subspace $U \subset V$ with a basis $u = u_1, u_2, \dots, u_m$ can be described by the $k \times m$ matrix A_u formed by the coordinate rows of vectors u_i in the chosen basis of V. Every other basis $w_1, w_2, \dots, w_m$ in U has the form $(w_1, w_2, \dots, w_m) = (u_1, u_2, \dots, u_m) \cdot C_{uw}$, where $C_{wu} \in \mathrm{GL}_k(\Bbbk)$, and leads to the matrix $A_w = C_{uw}^t A_u$.

Exercise 5.6. Check this.

Thus, two $k \times m$ matrices A_u, A_w of rank k correspond to the same subspace $U \subset V$ if and only if $A_w = GA_u$ for some $k \times k$ matrix $G \in \mathrm{GL}_k(\Bbbk)$. For $k = 1$, this agrees with the description of $\mathbb{P}_{m-1} = \mathrm{Gr}(1, m)$ as the set of nonzero rows $(x_1, x_2, \dots, x_m) \in \Bbbk^m = \mathrm{Mat}_{1\times m}$ considered up to multiplication by nonzero constants $\lambda \in \Bbbk^* = \mathrm{GL}_1(\Bbbk)$. Thus, the matrix $A_u \in \mathrm{Mat}_{k\times m}$, formed by coordinate rows of some basis vectors $u_1, u_2, \dots, u_k \in U$ and considered up to the left multiplication by matrices $G \in \mathrm{GL}_k$, is the direct analog of homogeneous coordinates on the projective space.

The Plücker embedding $\mathfrak{u} : \mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\Lambda^k V)$ takes a subspace $U \subset V$ of dimension k to the subspace $\Lambda^m U \subset \Lambda^m V$ of dimension 1. For every basis $u_1, u_2, \dots, u_m$ in U, the grassmannian monomial $u_1 \wedge u_2 \wedge \cdots \wedge u_m$ spans $\mathfrak{u}(U)$.

Exercise 5.7 (Plücker coordinates). Verify that for every $I = (i_1, i_2, \dots, i_k)$, the coefficient α_I in the expansion $u_1 \wedge u_2 \wedge \cdots \wedge u_m = \sum_I \alpha_I e_I$ equals the $k \times k$ minor situated in the columns $i_1, i_2, \dots, i_k$ of matrix A_u.

Thus, the $\binom{m}{k}$ homogeneous coordinates of the point $\mathfrak{u}(U) \in \mathbb{P}(\Lambda^k V)$ with respect to the basis formed by the grassmannian monomials e_I are the determinants $a_I = \det A_I$ of $k \times k$ submatrices $A_I \subset A_u$. They called the *Plücker coordinates* of the subspace $U \subset V$. Two subspaces $U, W \subset V$ of dimension k coincide if and only if their Plücker coordinates are proportional.

Exercise 5.8. Is there a rational 2×4 matrix with minors a) 2, 3, 4, 5, 6, 7 b) 3, 4, 5, 6, 7, 8? If such matrices exist, write some of them explicitly. If not, explain why.

5.2.1 Affine charts. For every subspace $T \subset V$ of codimension k, the set

$$\mathcal{U}_T \overset{\text{def}}{=} \{W \subset V \mid \dim W = k\,,\ W \cap T = 0\}$$

is called the *affine chart* provided by T on the grassmannian $\mathrm{Gr}(k, V)$. For every $U \in \mathcal{U}_T$, the set $\mathcal{U}_T$ is naturally identified with the affinization $\mathbb{A}\big(\mathrm{Hom}(U, T)\big)$ of the vector space of linear maps $\tau : U \to T$ as follows. We have the direct sum decomposition $V = T \oplus U$ and $\mathcal{U}_T$ consists of all those subspaces $W \subset U$ isomorphically projected onto U along T. Thus, every $W \in \mathcal{U}_T$ is the graph of linear map $\tau_W : U \to T$ sending a vector $u \in U$ to the unique vector $t \in T$ such that $u + t \in W$, and vice versa, for every linear map $\tau : U \to T$, its graph $W_\tau = \{u + \tau(u) \mid u \in U\}$ is a linear subspace in V isomorphically projected onto U along T.

For every $U \in \mathcal{U}_T$, the projection $V \twoheadrightarrow T$ along U assigns the isomorphism $\pi_T : V/U \xrightarrow{\sim} T$. It provides us with the linear isomorphism $\alpha_T : \mathrm{Hom}(U, V/U) \xrightarrow{\sim} \mathrm{Hom}(U, T)$, $\tau \mapsto \pi_T \circ \tau$, which allows to consider all affine charts $\mathcal{U}_T$ containing a given point $U \in \mathrm{Gr}(k, V)$ as affine spaces over the same vector space $\mathrm{Hom}(U, V/U)$ independent on T. Thus, locally, in a neighborhood of every point U, the grassmannian $\mathrm{Gr}(k, V)$ looks as an affine space over the vector space $\mathrm{Hom}(U, V/U)$ of dimension $k \times (m-k)$. This vector space is called the *tangent space* to the grassmannian $\mathrm{Gr}(k, V)$ at the point U and is denoted by $\mathcal{T}_U \mathrm{Gr}(k, V)$.

EXAMPLE 5.1 (AFFINE CHARTS ON $\mathbb{P}_{m-1} = \mathrm{Gr}(1, m)$ REVISITED)
Every codimension 1 subspace $T \subset V$ has the form $T = \mathrm{Ann}\,\xi$ for a non-zero covector $\xi \in V^*$ uniquely up to proportionality determined by T. Defined in n° 1.2 on p. 4 were affine charts U_ξ on $\mathbb{P}_{m-1} = \mathbb{P}(V)$. For all ξ such that $\mathrm{Ann}\,\xi = T$, the charts U_ξ consist of the same points, the dimension 1 subspaces $\Bbbk \cdot u \subset V$ such that $u \notin T$. Exactly the same subspaces form the chart $\mathcal{U}_T$ on $\mathrm{Gr}(1, V)$. This chart is an affine space associated with the vector space $\mathrm{Hom}(\Bbbk, T) \simeq T$. A particular choice of dimension 1 subspace $\Bbbk \cdot u \in \mathcal{U}_T$ fixes the origin in this affine space. Under this choice, every dimension 1 subspace $\Bbbk \cdot w$ laying in $\mathcal{U}_T$, i.e., such that $\xi(w) \neq 0$, can be identified with the linear map $\tau_w : \Bbbk \cdot u \to \mathrm{Ann}\,\xi = T$, $u \mapsto w \cdot \xi(u)/\xi(w) - u$. Note that this map depends only on the subspaces $\Bbbk \cdot u$, $\Bbbk \cdot w$, and T in V but not on the choice of $u \in \Bbbk \cdot u$, $w \in \Bbbk \cdot w$, and $\xi \in \mathrm{Ann}\,T$.

5.2.2 The standard affine charts on Gr(k,m). For every collection I of increasing indexes $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$, write $E_I, E_{\hat{I}} \subset \Bbbk^m$ for the complementary subspaces spanned by the basis vectors e_i, $i \in I$, and e_j, $j \notin I$, respectively. The affine chart $\mathcal{U}_{E_{\hat{I}}}$, which consists of all dimension k subspaces $U \subset \Bbbk^m$ isomorphically projected onto E_I along $E_{\hat{I}}$, is called the *standard I-chart* on grassmannian $\mathrm{Gr}(k, m)$ and denoted by $\mathcal{U}_I$.

For every subspace $U \subset V$ laying in the chart $\mathcal{U}_I$, write $u^{(I)} = u_1^{(I)}, u_2^{(I)}, \dots, u_k^{(I)}$ for the basis of U projected along $E_{\hat{I}}$ to the basis $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ of E_I. The matrix $A^{(I)} \stackrel{\text{def}}{=} A_{u^{(I)}}$, formed by the coordinate rows of these vectors, has the identity $k \times k$ submatrix in the columns $i_1, i_2, \dots, i_k$. We conclude that among the matrices A_u representing a subspace $U \in \mathcal{U}_I$, there exists the unique matrix having the identity submatrix in I-columns. We write $A^{(I)}(U)$ for this matrix and use the $k(m-k)$ elements staying outside the I-columns of $A^{(I)}(U)$ as the *standard affine coordinates* of U in the chart $\mathcal{U}_I$.

Clearly, a point $U \in \mathrm{Gr}(k, m)$ represented by a matrix $A = A_u \in \mathrm{Mat}_{k \times m}(\Bbbk)$ lies in $\mathcal{U}_I$ if and only if the $k \times k$ submatrix $A_I \subset A$ situated in I-columns of A is invertible. In this case, $A^{(I)}(U) = A_I^{-1} A$. Thus, the standard chart $\mathcal{U}_I$ consists of those U whose Ith Plücker coordinate is not zero. The matrices $A^{(I)} = A^{(I)}(U)$ and $A^{(J)} = A^{(J)}(U)$ producing the local affine coordinates of a point $U \in \mathcal{U}_I \cap \mathcal{U}_J$ in the standard charts $\mathcal{U}_I$, $\mathcal{U}_J$ are related as $A^{(I)} = (A_I^{(J)})^{-1} A^{(J)}$. Hence, the standard affine coordinates of the same subspace $U \subset V$ in different charts are rational functions of each other.

EXERCISE 5.9. Make it sure that the standard affine charts and local affine coordinates on $\mathrm{Gr}(1, m) = \mathbb{P}_{m-1}$ are exactly those introduced in Example 1.2 on p. 7.

EXERCISE 5.10. If you had deal with differential (respectively, analytic[1]) geometry, check that real (respectively complex) grassmannians are smooth (respectively holomorphic) manifolds.

5.3 The cell decomposition for Gr(k,m). The Gaussian elimination method shows that every subspace $U \subset V$ admits a unique basis $u = u_1, u_2, \dots, u_m$ with the *reduced echelon* matrix A_u, i.e., the leftmost nonzero element in every row of A_u stays strictly to the right of such element in the

[1] Also known as holomorphic.

previous row, equals 1, and is the only nonzero element of its column.

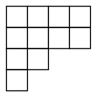
EXERCISE 5.11. Convince yourself that the rows of different reduced echelon $k \times m$ matrices span different subspaces in $\Bbbk^m$.

Thus, there exist a bijection between $\mathrm{Gr}(k, m)$ and the set of reduced echelon $k \times m$ matrices of rank m. The latter splits in disjoint union of affine spaces as follows. Write $J = j_1, j_2, \dots, j_k$ for successive numbers of those columns containing the starting units of rows in a reduced echelon matrix A, and call this increasing sequence of integers the *shape* of A. Every reduced echelon $k \times m$ matrix A of shape I contains the identity submatrix in the J-columns, and has exactly

$$k(m-k) - (j_1 - 1) - (j_2 - 2) - \cdots - (j_m - m) = \dim \mathrm{Gr}(k, m) - \sum_{\nu=1}^{m} (j_\nu - \nu)$$

free cells which may contain arbitrary elements of $\Bbbk$. Thus, these matrices form an affine space of codimension $\sum_{\nu=1}^{m} (j_\nu - \nu)$ in $\mathrm{Gr}(k, m)$. It is denoted by α_J and called an *affine Schubert cell*. The whole grassmannian splits in disjoint union of $\binom{m}{k}$ such cells: $\mathrm{Gr}(k, m) = \bigsqcup_J \alpha_J$.

5.3.1 Young diagram notations. Besides the strictly increasing sequences of integers, the *partitions* are also commonly used for indexing the Schubert cells. A *partition* λ is a non-increasing sequence of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$ visualized as the *Young diagram*, the pile of horizontal cellular strips of lengths $\lambda_1, \lambda_2, \dots, \lambda_m$ aligned to the left in the non-increasing top-down order. For example, the partition $(4, 4, 2, 1)$ has the Young diagram



(5-6)

The total number of cells in a diagram λ is denoted by $|\lambda| \overset{\text{def}}{=} \sum \lambda_i$ and called the *weight* of λ. Thus, the partitions of weight n enumerate the ways to break a set of n mutually elements in a union of disjoint subsets. The total number of non-empty parts is called the *height* of partition and denoted by $h(\lambda) = \max(k \mid \lambda_k > 0)$. The cardinality λ_1 of biggest part is called the *width* of the partition. For example, the diagram (5-6) has weight 11, height 4, and width 4.

We say that a reduced echelon matrix A has the shape λ for some partition $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k$ if for every $i = 1, 2, \dots, k$, the starting unit in the ith from the bottom row of A stays in the λ_ith cell to the right of the leftmost possible position. This means that $\lambda_{k+1-\nu} = j_\nu - \nu$ for every $\nu = k + 1 - i = 1, 2, \dots, k$. Note that the codimension of the affine Schubert cell α_λ equals the weight $|\lambda|$ of Young diagram λ.

EXERCISE 5.12. Convince yourself that the prescription $j_1, j_2, \dots, j_k \mapsto \lambda_1, \lambda_2, \dots, \lambda_k$ such that $\lambda_{k+1-\nu} = j_\nu - \nu$ for all $1 \leqslant \nu \leqslant k$ establishes a bijection between the sequences of k strictly increasing integers in range $[0, m]$ and the Young diagrams ot height $\leqslant k$ and width $\leqslant m - k$.

For example, the affine Schubert cell $\alpha_{4421} \subset \mathrm{Gr}(4, 10)$ corresponding to the diagram (5-6) consists of subspaces $U \subset \Bbbk^{10}$ represented by reduced echelon matrices of the shape

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

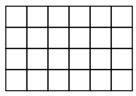
Colored in red are the leftmost possible positions for the starting units of reduced echelon 4×10 matrix. Colored in blue are the actual starting units. Being read bottom-up, they are sifted by 4, 4, 2, and 1 cell to the right of red cell. The grassmannian $\mathrm{Gr}(4,10)$ has dimension 24, the codimension of $\alpha_{4421} \simeq \mathbb{A}^{13}$ equals $11 = 4+4+2+1$.

The zero partition $(0,0,0,0)$ has empty Young diagram meaning that the starting units stay in the leftmost possible positions. It describes the largest Schubert cell α_0 of dimension 24 which consists of subspaces $U \subset \Bbbk^{10}$ represented by matrices of the shape

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \end{pmatrix}$$

Thus, the cell α_0 coincides with the standard affine chart $\mathcal{U}_{1234} \subset \mathrm{Gr}(4,10)$.

The maximal possible for $\mathrm{Gr}(4,10)$ Young diagram $(6,6,6,6)$ exhausts the whole rectangle



and describes one point cell, the coordinate subspace $E_{7,8,9,10} \subset \Bbbk^{10}$ spanned by the rows of matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

5.3.2 The closed Schubert cycles. We write $\lambda \subseteq \mu$ if the diagram λ is contained in the diagram μ sharing the same upper left corner. Consider a pair of such diagrams and a subspace $W \subset \Bbbk^m$ such that $W \in \alpha_\mu$ in $\mathrm{Gr}(k,m)$. Let A be the reduced echelon matrix of W, B the reduced echelon matrix of shape λ corresponding to the origin of affine cell α_λ, i.e., all element of B but the starting units of rows equal zero. For every $t = (t_0 : t_1) \in \mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ except for $t = (1:0)$, the reduced echelon form of matrix $C_t = t_0 A + t_1 B$ has the shape λ but $C_\infty = A$ is of shape μ. The subspace $U_t \subset \Bbbk^m$ spanned by the rows of matrix C_t draws a rationally parameterized curve in $\mathrm{Gr}(k,m) \subset \mathbb{P}(\Lambda^k \Bbbk^m)$ as t runs through $\mathbb{P}_1$. All points of this curve but $U_\infty = W \in \alpha_\mu$ belong to the affine Schubert cell α_λ. We conclude that the affine cell α_μ lies in the closure of α_λ for all $\mu \supseteq \lambda$. For every Young diagram λ contained in the rectangle $k \times (m-k)$, the union $\sigma_\lambda = \bigsqcup_{\mu \supseteq \lambda} \alpha_\mu$ is called the (*closed*) *Schubert cycle* of grassmannian $\mathrm{Gr}(k,m)$.

Write $E_{\geqslant n} \subset \Bbbk^m$ for the coordinate subspace spanned by $e_n, e_{n+1}, \dots, e_m$, and $E_{<n}$ for the complementary coordinate subspace. Then, in J-notations, σ_J consists of those subspaces $U \subset \Bbbk^m$ mapped by the projection $\pi_\nu : \Bbbk^m \twoheadrightarrow E_{<j_\nu}$ along $E_{\geqslant j_\nu}$ to a subspace of dimension $\leqslant \nu - 1$ for every $1 \leqslant \nu \leqslant k$, or equivalently, of those U intersecting $\ker \pi_\nu = E_{\geqslant j_\nu}$ in a subspace of dimension at least $k+1-\nu$. Thus, $\sigma_J = \{U \subset \Bbbk^m \mid \dim(U \cap E_{\geqslant j_\nu}) \geqslant k+1-\nu \text{ for } \nu = 1,\dots,k\}$. This is translated in λ-notations as $\sigma_\lambda = \{U \subset \Bbbk^m \mid \dim(U \cap E_{\geqslant k+1-i+\lambda_i}) \geqslant i \text{ for } i = 1,\dots,k\}$.

EXERCISE 5.13. Convince yourself that for $\Bbbk = \mathbb{R}, \mathbb{C}$, the Schubert cycles are closed submanifolds of the grassmannian $\mathrm{Gr}(k,m)$.

EXAMPLE 5.2 (THE SCHUBERT CELLS ON $\mathrm{Gr}(2,4)$)
In $\mathbb{P}_3 = \mathbb{P}(\Bbbk^4)$, consider the point $a = (0:0:0:1)$ and plane $\Pi = V(x_0)$. Then the strata of stratification from formula (5-4) on p. 60 are the Plücker images of Schubert cycles on $\mathrm{Gr}(2,4)$.

Namely, in the notations of n° 5.1.2, the α-plane $\pi_\alpha(a)$ on the Plücker quadric $P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2\Bbbk^4)$ is the Plücker image of Schubert cycle σ_{20}, i.e., the closure of affine cell α_{11} formed by reduced echelon matrices $\left(\begin{smallmatrix}1&*&*&0\\0&0&0&1\end{smallmatrix}\right)$. The β-plane $\pi_\beta(\Pi)$ is the cycle σ_{11}, the closure of affine cell α_{20} formed by matrices $\left(\begin{smallmatrix}0&1&0&*\\0&0&1&*\end{smallmatrix}\right)$. Their intersection $\pi_\alpha(a) \cap \pi_\beta(\Pi) = (pp')$ equals σ_{21}, the closure $\alpha_{21} \sqcup \alpha_{22}$ of the cell α_{21} formed by matrices of shape $\left(\begin{smallmatrix}0&1&*&0\\0&0&0&1\end{smallmatrix}\right)$. The dimension zero cycle $\sigma_{22} = \alpha_{22}$ is the point $p = (0 : 0 : 0 : 0 : 0 : 1) \in \mathbb{P}_5 = \mathbb{P}(\Lambda^2\Bbbk^4)$, the Plücker image of matrix $\left(\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right)$. The cone $T_p \cap P = \sigma_{10}$ is the closure of α_{10}, the affine cell formed by matrices $\left(\begin{smallmatrix}1&*&0&*\\0&0&1&*\end{smallmatrix}\right)$. The biggest cycle σ_{00} is the whole Plücker quadric P.

EXERCISE 5.14. Check all these statements carefully.

5.3.3 The homology of complex grassmannians and Schubert calculus.

Write

$$\Lambda(k,m) \overset{\text{def}}{=} \bigoplus_i H_i\big(\mathrm{Gr}(k,\mathbb{C}^m),\mathbb{Z}\big)$$

for the total integer homology group of the complex grassmannian considered as a (real) topological manifold. The (open) affine Schubert cells α_λ provide $\mathrm{Gr}(k,m)$ with the cell decomposition which consists of even dimensional cells only. Hence, all boundary maps in the chain complex constructed by means of this chain decomposition vanish. Therefore, the closed Schubert cycles $\sigma_\lambda = \overline{\alpha}_\lambda$ form a basis of $\Lambda(k,m) = \bigoplus_i H_i$ over $\mathbb{Z}$. E.g., for the Plčker quadric $P = \mathrm{Gr}(2,\mathbb{C}^4) \subset \mathbb{P}(\mathbb{C}^5)$ of real dimension 8, we have $H_0 = H_2 = H_6 = H_8 = \mathbb{Z}$, $H_4 = \mathbb{Z} \oplus \mathbb{Z}$, and all the homology of odd dimension vanishes. This agrees with Exercise 5.4 on p. 62.

Topological intersection of cycles provides $\Lambda(k,m)$ with the structure of commutative ring closely connected with the ring Λ_m of symmetric polynomials in m variables, which is the polynomial ring $\Lambda_m = \mathbb{Z}[\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m] \subset \mathbb{Z}[x_1,x_2,\dots,x_m]$ generated by the *elementary* symmetric polynomials[1] $\varepsilon_k(x_1,x_2,\dots,x_m)$. Namely, there is the surjective homomorphism of commutative rings $\Lambda_m \twoheadrightarrow \Lambda(k,m)$ sending the *Schur polynomial*[2] $s_\lambda(x_1,x_2,\dots,x_m)$ to the Schubert cycle σ_λ. The kernel ideal of this homomorphism is spanned by *complete* symmetric polynomials[3] $\eta_{m-k+1},\dots,\eta_m$ of degrees in range $[m-k+1,m]$. All known[4] proofs of these statements are indirect and besides the

[1]Recall that ε_n is sum of all *multilinear* monomials of total degree n in $x_1,x_2,\dots,x_m$.

[2]The Schur polynomial $s_\lambda \in \mathbb{Z}[x_1,x_2,\dots,x_m]$ is defined either as the quotient of determinant

$$\Delta_\lambda = \det(x_j^{\lambda_i+m-i}) = \det\begin{pmatrix} x_1^{\lambda_1+m-1} & x_2^{\lambda_1+m-1} & \cdots & x_m^{\lambda_1+m-1} \\ x_1^{\lambda_2+m-2} & x_2^{\lambda_2+m-2} & \cdots & x_m^{\lambda_2+m-2} \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{\lambda_{m-1}+1} & x_2^{\lambda_{m-1}+1} & \cdots & x_n^{\lambda_{m-1}+1} \\ x_1^{\lambda_m} & x_2^{\lambda_m} & \cdots & x_n^{\lambda_m} \end{pmatrix}$$

by the Vandermonde determinant $\Delta_{0,\dots,0}$ or as the sum of all monomials in $x_1,x_2,\dots,x_m$ obtained as follows: fill the cells of diagram λ by (possibly repeated) variables $x_1,x_2,\dots,x_m$ in such a way that indexes strictly increase top-down in columns and non-strictly increase from left to right in rows, then multiply them altogether to one monomial of total degree $|\lambda|$. E.g, for the one-column diagram of height h, we get $s_{1,1,\dots,1} = \varepsilon_h$. The coincidence of two descriptions is non-trivial and known as the *Jacobi – Trudi identity*. For details, see W. Fulton, *Young Tableaux with Applications to Representation Theory and Geometry*, CUP, 1997.

[3]Recall that the complete symmetric polynomial η_n equals the sum of all degree n monomials in $x_1,x_2,\dots,x_m$ at all.

[4]At least, to me.

geometry of grassmannians, use sophisticated combinatorics of symmetric functions. The geometric part of the proof establishes two basic intersection rules:

1) The intersection of cycles σ_λ, σ_μ of complementary codimensions $|\lambda|+|\mu| = k(m-k)$ is not zero if and only if the diagrams λ, μ are complementary[1], and in this case, the intersection consists of one point, that is, equals $\sigma_{k,\dots,k}$.

2) The *Pieri rules*: for any integer n and diagram λ, $\sigma_\lambda\sigma_{(n,0,\dots,0)} = \sum \sigma_\mu$ and $\sigma_\lambda\sigma_{\underbrace{(1,\dots,1)}_n} = \sum \sigma_\nu$, where μ, ν run through the Young diagrams obtained by adding n cells to λ in such a way that all added cells appear in different rows of μ and in different columns of ν. If there are no such diagrams, the intersection is zero.

The proofs can be found, e.g., in: P. Griffits, J. Harris, *Principles of Algebraic Geometry, I*. It follows from the determinantal definition of Schubert polynomials that they form a basis over $\mathbb{Z}$ in the additive group of symmetric polynomials, because the alternating sums

$$\Delta_\lambda = \det(x_j^{\lambda_i+m-i}) = \sum_{g\in S_m} \operatorname{sgn}(g) x_{g(1)}^{\lambda_1+m-1} x_{g(2)}^{\lambda_2+m-2} \dots x_{g(m)}^{\lambda_m}$$

obviously form a basis in the additive group of alternating polynomials in $x_1, x_2, \dots, x_m$, and dividing by the Vandermonde determinant maps this group isomorphically to the additive group of symmetric polynomials.

Exercise 5.15. Show that every alternating polynomial in $x_1, x_2, \dots, x_m$ is divisible by the Vandermonde determinant in the polynomial ring $\mathbb{Z}[x_1, x_2, \dots, x_m]$.

The combinatorial part of the proof verifies that the multiplication of Schur polynomials also satisfies the Pieri rules, which are particular cases of the Littlewood – Richardson rule for multiplying arbitrary Schur polynomials[2]. It is easy to see that the Pieri rules completely determine the multiplicative structure in the both rings Λ_m, $\Lambda(k,m)$. This proves that the map $\Lambda_m \to \Lambda(k,m)$, $s_\lambda \mapsto \sigma_\lambda$, is a well defined surjective homomorphism of rings. The description of its kernel comes from the intersection rule (1) for the Schubert cycles of complementary dimensions. The details of this story, known as the *Schubert calculus*, can be found in the cited book of P. Griffits and J. Harris and in the *Intersection Theory* book of W. Fulton.

Example 5.3 (the intersection theory on Gr(2, 4))

As we have seen in Example 5.2, the Schubert cycles on Gr(2, 4) can be realized as

$\sigma_{10}(\ell) = \{\ell' \subset \mathbb{P}_3 \mid \ell \cap \ell' \neq \varnothing\}$ for a line $\ell \subset \mathbb{P}_3$,
$\sigma_{20}(a) = \{\ell' \subset \mathbb{P}_3 \mid \ell' \ni a\}$ for a point $a \in \mathbb{P}_3$,
$\sigma_{11}(\Pi) = \{\ell' \subset \mathbb{P}_3 \mid \ell' \subset \Pi\}$ for a plane $\Pi \subset \mathbb{P}_3$,
$\sigma_{21}(a,\Pi) = \sigma_{20}(a) \cap \sigma_{11}(\Pi)$ for $a \in \Pi \subset \mathbb{P}_3$,
$\sigma_{22}(\ell) = \{\ell\}$, a line $\ell \subset \mathbb{P}_3$ considered as a point of Gr(2, 4).

Certainly, $\sigma_{ij}\sigma_{k\ell} = 0$ for $i+j+k+\ell = \operatorname{codim}\sigma_{ij} + \operatorname{codim}\sigma_{k\ell} > 4$. We have seen in Example 5.2 that $\sigma_{20}(a_1) \cap \sigma_{20}(a_2) = \sigma_{22}((a_1a_2))$, $\sigma_{11}(\Pi_1) \cap \sigma_{11}(\Pi_2) = \sigma_{22}(\Pi_1 \cap \Pi_2)$, whereas for $a \notin \Pi$,

[1] That is, can be fitted together without holes and overlaps to assemble $k \times (m-k)$ rectangle.

[2] See already cited W. Fulton's book on Young diagrams, or Sec. 4.5 in: A.L.Gorodentsev, *Algebra II. Textbook for Students of Mathematics*, Springer, 2017. The Pieri rules can be proven independently on the Littlewood – Richardson rule by formal algebraic manipulations with determinants, see, e.g., Section 3.6 of *loc. cit.*

$\sigma_{20}(a) \cap \sigma_{11}(\Pi) = \varnothing$. By the same geometric reasons, for a line ℓ and a plane Π intersecting at a point b, we have $\sigma_{10}(\ell) \cap \sigma_{11}(\Pi) = \sigma_{21}(b, \Pi)$. Dually, for a line ℓ and a point $a \notin \ell$, we have $\sigma_{10}(\ell) \cap \sigma_{20}(a) = \sigma_{21}(a, \Pi)$, where Π is the plane passing trough a and ℓ. Similarly, for a point a in a plane Π, and a line ℓ intersecting Π in a point $b \neq a$, we get $\sigma_{10}(\ell) \cap \sigma_{21}(a, \Pi) = \sigma_{22}((a, b))$. For a generic choice of lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ the intersection $\sigma_{10}(\ell_1) \cap \sigma_{10}(\ell_2)$, which consists of all lines intersecting both ℓ_1, ℓ_2, is the Segre quadric laying in $\mathbb{P}_3 = T_{\mathrm{u}(\ell_1)}P \cap T_{\mathrm{u}(\ell_1)}P$ as it was shown in fig. 5◇1 on p. 61. However, when the lines ℓ_1, ℓ_2 are intersecting but still different, the intersection $\sigma_{10}(\ell_1) \cap \sigma_{10}(\ell_2)$ splits in the union of the α-net $\sigma_{20}(a)$ centered at the intersection point $a = \ell_1 \cap \ell_2$ and the β-net $\sigma_{11}(\Pi)$, where Π is the plane containing ℓ_1, ℓ_2. Since the integer homology classes of all cycles just mentioned are not changed under continuous moving of the points, lines, and planes in $\mathbb{P}_3$ used to construct the realizations of these cycles within $\mathrm{Gr}(2,4)$, we conclude that nonzero products of the Schubert cycles in $\mathrm{Gr}(2,4)$ are exhausted by

$$\sigma_{10}^2 = \sigma_{20} + \sigma_{11}\,, \quad \sigma_{10}\sigma_{20} = \sigma_{10}\sigma_{11} = \sigma_{21}\,, \quad \sigma_{10}\sigma_{21} = \sigma_{20}^2 = \sigma_{11}^2 = \sigma_{22}\,,$$

and $\sigma_{00}\sigma_{ij} = \sigma_{ij}$ for all Young diagrams (ij) went in the square 2×2. As a byproduct, we get a «topological» solution of Exercise 2.14 on p. 23: for a generic choice of 4 mutually non-intersecting lines in $\mathbb{P}_3$, the set of lines intersecting them all represents the homology class of topological fourfold self-intersection $\sigma_{10}^4 = (\sigma_{20} + \sigma_{11})^2 = \sigma_{20}^2 + \sigma_{11}^2 = 2\sigma_{22}$, that is, consists of two lines.

§6 Commutative algebra draught

Everywhere in §6, the term «ring» means by default a commutative ring with unit. All ring homomorphisms are assumed to map the unit to the unit.

6.1 Noetherian rings. Every subset M in a commutative ring K generates an ideal $(M) \subset K$ formed by all finite sums $b_1a_1+b_2a_2+\cdots+b_ma_m$, where $a_1, a_2, \dots, a_m \in M$, $b_1, b_2, \dots, b_m \in K$, $m \in \mathbb{N}$. Every ideal $I \subset K$ is generated by some subset $M \subset K$, e.g., by $M = I$. An ideal $I \subset M$ is said to be *finitely generated* if it admits a finite set of generators, that is, if it can be written as $I = (a_1, a_2, \dots, a_k) = \{b_1a_1 + b_2a_2 + \cdots + b_ka_k \mid b_i \in K\}$ for some $a_1, a_2, \dots, a_k \in I$.

Lemma 6.1
The following properties of a commutative ring K are equivalent:

1) Every subset $M \subset K$ contains some finite collection of elements $a_1, a_2, \dots, a_k \in M$ such that $(M) = (a_1, a_2, \dots, a_k)$.

2) Every ideal $I \subset K$ is finitely generated.

3) For every infinite chain of increasing ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ in K there exists $n \in \mathbb{N}$ such that $I_\nu = I_n$ for all $\nu \geqslant n$.

Proof. Clearly, (1) ⇒ (2). To deduce (3) from (2), write $I = \bigcup I_\nu$ for the union of all ideals in the chain. Then I is an ideal as well. By (2), I is generated by some finite set of its elements. All these elements belong to some I_n. Therefore, $I_n = I = I_\nu$ for all $\nu \geqslant n$. To deduce (1) from (3), we construct inductively a chain of strictly increasing ideals $I_n = (a_1, a_2, \dots, a_n)$ starting from an arbitrary $a_1 \in M$. While $I_k \neq (M)$, we choose any element $a_{k+1} \in M \smallsetminus I_k$ and put $I_{k+1} = (a_{k+1} \cup I_k)$. Since $I_k \subsetneq I_{k+1}$ in each step, by (3) this procedure has to stop after a finite number of steps. At that moment, we obtain $I_m = (a_1, a_2, \dots, a_m) = (M)$. □

Definition 6.1
A commutative ring K is called to be *Noetherian* if it satisfies the conditions from Lemma 6.1. Note that every field is Noetherian.

Theorem 6.1 (Hilbert's basis theorem)
For every Noetherian commutative ring K the polynomial ring $K[x]$ is Noetherian as well.

Proof. Consider an arbitrary ideal $I \subset K[x]$ and write $L_d \subset K$ for the set of leading coefficients of all polynomials of degree $\leqslant d$ in I including the zero polynomial. Also we write $L_\infty = \cup_d L_d$ for the set of all leading coefficients of all polynomials in I.

Exercise 6.1. Verify that all of the L_d and L_∞ are the ideals in K.

Since K is Noetherian, all ideals L_d and L_∞ are finitely generated. For all d (including $d = \infty$), write $f_1^{(d)}, f_2^{(d)}, \dots, f_{m_d}^{(d)} \in K[x]$ for those polynomials whose leading coefficients span the ideal $L_d \subset K$. Let $D = \max \deg f_i^{(\infty)}$. We claim that polynomials f_i^{∞} and $f_j^{(d)}$ for $d < D$ generate I. Let us show first that each polynomial $g \in I$ is congruent modulo $f_1^{(\infty)}, f_2^{(\infty)}, \dots, f_{m_\infty}^{(\infty)}$ to some polynomial of degree less than D. Since the leading coefficient of g lies in L_∞, it can be written as $\sum \lambda_i a_i$, where $\lambda_i \in K$ and a_i is the leading coefficient of $f_i^{(\infty)}$. As long as $\deg g \geqslant D$ all differences $m_i = \deg g - \deg f_i^{(\infty)}$ are nonnegative, and we can form the polynomial $h = g - \sum \lambda_i \cdot f_i^{(\infty)}(x) \cdot x_i^{m_i}$, which is congruent to g modulo I and has $\deg h < \deg g$. We replace g by h and repeat the procedure while $\deg h \geqslant D$. When we come to a polynomial $h \equiv g \pmod{I}$ such that $\deg h < D$, the leading coefficient of h falls into some L_d with $d < D$, and we can cancel the leading terms of h by subtracting appropriate combinations of polynomials $f_j^{(d)}$ for $0 \leqslant d < D$ until we get $h = 0$. □

COROLLARY 6.1
For every Noetherian commutative ring K, the ring $K[x_1, x_2, \dots, x_n]$ is Noetherian. □

EXERCISE 6.2. For every Noetherian commutative ring K show that the ring $K[\![x_1, x_2, \dots, x_n]\!]$ of formal power series in $x_1, x_2, \dots, x_n$ with coefficients in K is Noetherian as well.

COROLLARY 6.2
Every infinite system of polynomial equations with coefficients in a Noetherian commutative ring K is equivalent to some finite subsystem.

PROOF. Since $K[x_1, x_2, \dots, x_n]$ is Noetherian, among the right hand sides of a polynomial equation system $f_\nu(x_1, x_2, \dots, x_n) = 0$ there is some finite collection $f_1, f_2, \dots, f_m$ that generates the same ideal as all the f_ν. This means that every $f_\nu = g_1 f_1 + g_2 f_2 + \dots + g_m f_m$ for some $g_i \in K[x_1, x_2, \dots, x_n]$. Hence, every equation $f_\nu = 0$ follows from $f_1 = f_2 = \dots = f_m = 0$. □

EXERCISE 6.3. Show that all quotient rings of a Noetherian ring are Noetherian.

CAUTION 6.1. A subring of a Noetherian ring is not necessary Noetherian. For example, the ring $\mathbb{C}[\![z]\!]$ is Noetherian by Exercise 6.2. However, the subring $\mathcal{H} \subset \mathbb{C}[\![z]\!]$ of holomorphic functions[1] $f : \mathbb{C} \to \mathbb{C}$ is not Noetherian, because there exist a sequence of holomorphic functions $f_n : \mathbb{C} \to \mathbb{C}$ such that for all $n \in \mathbb{N}$, $f_n(z) = 0$ exactly for $z \in \mathbb{Z} \smallsetminus [-n, n]$ and therefore, $I_n = (f_1, f_2, \dots, f_n)$ form an infinite chain of strictly increasing ideals.

EXERCISE 6.4. Construct such a sequence $(f_n)_{n \in \mathbb{N}}$ explicitly.

6.2 Integral elements. An *extension of rings* is a pair $A \subset B$, where A is a subring of a ring B and both rings have common unit. Given such a ring extension $A \subset B$, an element $b \in B$ is called *integral* over A if it satisfies the conditions of the following lemma.

LEMMA 6.2 (CHARACTERIZATION OF INTEGRAL ELEMENTS)
The following properties of an element $b \in B$ in a ring extension $A \subset B$ are equivalent:

(1) $b^m = a_1 b^{m-1} + \dots + a_{m-1} b + a_m$ for some $m \in \mathbb{N}$ and $a_1, a_2, \dots, a_m \in A$.

(2) The A-linear span of all nonnegative integer powers b^m is a finitely generated A-module.

(3) There exists a finitely generated A-module $M \subset B$ such that $bM \subset M$ and $b'M \neq 0$ for all nonzero $b' \in B$.

PROOF. The implications (1) ⇒ (2) ⇒ (3) are obvious. Let us show that (3) ⇒ (1). Fix some $e_1, e_2, \dots, e_m$ spanning M over A. Then $(be_1, be_2, \dots, be_m) = (e_1, e_2, \dots, e_m) \cdot Y$ for some matrix

[1] That is, power series converging everywhere in $\mathbb{C}$.

$Y \in \mathrm{Mat}_m(A)$ and therefore, $(e_1, e_2, \dots, e_m) \cdot (bE - Y) = 0$. It follows from the matrix identity[1] $\det X \cdot E = X \cdot X^\vee$, where X is a square matrix over a commutative ring, E is the identity matrix of the same size, and $X^\vee$ is the adjunct matrix[2] of X, that the image of multiplication by $\det X$ lies in the linear span of the columns of the matrix X. For $X = (bE - Y) \in \mathrm{Mat}_m(B)$, this means that $\det(bE - Y) \cdot M$ is contained in the B-linear span of vectors $(e_1, e_2, \dots, e_m) \cdot (bE - Y)$, which is zero. The last property in (3) forces $\det(bE - Y) = 0$. Since all elements of Y lie in A, the latter equality can be rewritten in the form appearing in (1). □

DEFINITION 6.2
Let $A \subset B$ be an extension of rings. The set of all elements $b \in B$ integral over A is called the *integral closure* of A in B. If it coincides with A, then A is said to be *integrally closed* in B. If all elements of B are integral over A, then the extension $A \subset B$ is called an *integral* ring extension, and we say that B is *integral* over A.

EXAMPLE 6.1 ($\mathbb{Z}$ IS INTEGRALLY CLOSED IN $\mathbb{Q}$)
Let $A = \mathbb{Z}$, $B = \mathbb{Q}$. If a fraction $p/q \in \mathbb{Q}$ with coprime $p, q \in \mathbb{Z}$ satisfies a monic polynomial equation

$$\frac{p^m}{q^m} = a_1 \frac{p^{m-1}}{q^{m-1}} + \cdots + a_{m-1} \frac{p}{q} + a_m$$

with $a_i \in \mathbb{Z}$, then $p^m = a_1 q p^{m-1} + \cdots + a_{m-1} q^{m-1} p + a_m q^m$ is divisible by q. Since p, q are coprime, we conclude that $q = \pm 1$. Hence, $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

EXAMPLE 6.2 (INVARIANTS OF A FINITE GROUP)
Let a finite group G act on a ring B by ring automorphisms, and $B^G \stackrel{\text{def}}{=} \{ a \in B \mid ga = a \ \forall g \in G \}$ be the subring of G-*invariants*. Then B is integral over B^G. Indeed, write $b_1, b_2, \dots, b_n$ for the G-orbit of an arbitrary element $b = b_1 \in B$. Then b is a root of the monic polynomial

$$f(t) = \prod (t - b_i) \in B^G[t]$$

as required in the first property of Lemma 6.2.

PROPOSITION 6.1
Let $A \subset B$ be an extension of rings, and $\overline{A}_B \subset B$ the integral closure of A in B. Then $\overline{A}_B$ is a subring of B, and for any ring extension $C \supset B$, every element $c \in C$ integral over $\overline{A}_B$ is integral over A as well.

PROOF. If elements $p, q \in B$ satisfy the monic polynomial equations

$$p^m = x_1 p^{m-1} + \cdots + x_{m-1} p + x_m$$
$$q^n = y_1 q^{n-1} + \cdots + y_{n-1} q + y_n$$

for some $x_\nu, y_\mu \in A$, then the products $p^i q^j$ with $0 \leqslant i < m - 1$, $0 \leqslant j < n - 1$ span a finitely generated A-module, containing the unit and mapped to itself by the multiplication by p and by q.

[1] This is $n = 1$ case of the Laplace identity $\mathcal{X}_n \cdot \mathcal{X}_n^\vee = \det X \cdot E$ from the Example 4.4 on p. 46.
[2] That is, transposed to the matrix of algebraic complements $(-1)^{i+j} x_{\hat{i}\hat{j}}$ to the elements x_{ij} of matrix X, see Example 4.4 on p. 46.

Therefore, it satisfies the condition (3) from Lemma 6.2 for both $b = p + q$ and $b = pq$. Similarly, if the monic polynomial equations

$$c^r = z_1 c^{r-1} + \cdots + z_{r-1} c + z_r$$
$$z_k^{m_k} = a_{k,1} z^{m_k-1} + \cdots + a_{k,m_k-1} z_k + a_{k,m_k} \quad 1 \leqslant k \leqslant r,$$

hold for some $c \in C$, $z_1, z_2, \dots, z_r \in \overline{A}_B$, and $a_{k,\ell} \in A$, then the A-linear span of products

$$c^i z_1^{j_1} z_2^{j_2} \cdots z_r^{j_r}, \quad 0 \leqslant i < r-1,\ 0 \leqslant j_k < m_k - 1,$$

contains the unit and goes to itself under the multiplication by c. Thus, c is integral over A. □

PROPOSITION 6.2 (GAUSS – KRONECKER – DEDEKIND LEMMA)
Let $A \subset B$ be an extension of rings, and $f, g \in B[x]$ monic polynomials of positive degree. Then all coefficients of the product fg are integral over A if and only if all coefficients of the polynomials f, g are integral A.

PROOF. Let $C \supset B$ be an extension of rings such that the polynomials f, g are completely factorisable in $C[x]$ as $f(x) = \prod(x - \alpha_\nu)$ and $g(x) = \prod(x - \beta_\mu)$ for some $\alpha_\nu, \beta_\mu \in C$. Then their product $h(x) = f(x)g(x) = \prod(x - \alpha_\nu)\prod(x - \beta_\mu)$ is also completely factorisable.

EXERCISE 6.5. Given a finite set of monic polynomials of positive degree in $B[x]$, prove that there is an extension of rings $B \subset C$ such that all polynomials become completely factorisable in $C[x]$.

If all coefficients of h are integral over A, then all the roots $\alpha_\nu, \beta_\mu \in C$ are integral over $\overline{A}_C$ and therefore integral over A by Proposition 6.1. Since integral elements form a ring, all coefficients of both f, g, which are the symmetric functions of α_ν, β_μ, are also integral over A. The same arguments work in the opposite direction as well. □

PROPOSITION 6.3
Let $A \subset B$ be an integral extension of rings. If B is a field, then A is a field too. Conversely, if A is a field and B has no zero divisors, then B is a field.

PROOF. Let B be an integral field over A. Then, for any nonzero $a \in A$, the inverse element $a^{-1} \in B$ satisfies a monic polynomial equation $a^{-m} = \alpha_1 a^{1-m} + \cdots + \alpha_{m-1} a^{-1} + \alpha_m$ for some $\alpha_\nu \in A$. Multiplication of the both sides by a^{m-1} shows that $a^{-1} = \alpha_1 + \alpha_2 a + \cdots + \alpha_m a^{m-1} \in A$.

Conversely, if B is an integral algebra over a field A, then for every $b \in B$, the A-linear span of all nonnegative integer powers b^m is a vector space V of finite dimension over A. If $b \neq 0$, the linear endomorphism $b : V \to V$, $x \mapsto bx$, is injective, because B has no zero divisors. This forces it to be bijective. The preimage of the unit $1 \in V$ is b^{-1}. □

6.3 Normal rings. A commutative ring A without zero divisors is called *normal* if A is integrally closed in its field of fractions Q_A. In particular, every field is normal. The same arguments as in Example 6.1 show that every unique factorization domain A is normal. Indeed, a polynomial $a_0 t^m + a_1 t^{m-1} + \cdots + a_{m-1} t + a_m \in A[t]$ annihilates a fraction $p/q \in Q_A$ with $(p, q) = 1$ only if $q \mid a_0$ and $p \mid a_m$. Therefore, $a_0 = 1$ forces $q = 1$. As a consequence, the polynomial rings over a unique factorization domain are normal. For normal rings, Proposition 6.2 leads to the following classical claim going back to Gauss.

Corollary 6.3 (Gauss lemma II)

Let A be a normal ring, Q_A its field of fractions, and $f \in A[x]$ a monic polynomial. If $f = gh$ in $Q_A[x]$ for some monic polynomials g, h, then $f, g \in A[x]$. □

Corollary 6.4

Under the conditions of Corollary 6.3, let $B \supset Q_A$ be a ring extending Q_A. If an element $b \in B$ is integral over A, then the minimal polynomial[1] of b over Q_A lies in $A[x]$.

Proof. Since b is integral over A, there exists a monic polynomial $f \in A[x]$ such that $f(b) = 0$. The minimal polynomial of b over Q_A divides f in $Q_A[x]$, and the quotient is also monic. It remains to apply Corollary 6.3. □

6.4 Algebraic elements. Let B be a commutative algebra with unit over an arbitrary field $\Bbbk$. Given an element $b \in B$, we write $\Bbbk[b] \subset B$ for the smallest $\Bbbk$-subalgebra containing 1 and b. It coincides with the image of evaluation map

$$\mathrm{ev}_b : \Bbbk[x] \to B, \quad f \mapsto f(b). \tag{6-1}$$

Recall that b is said to be *transcendental* over $\Bbbk$ if $\ker \mathrm{ev}_b = 0$. In this case, $\Bbbk[b] \simeq \Bbbk[x]$ is infinite-dimensional as a vector space over $\Bbbk$ and is not a field. If $\ker \mathrm{ev}_b \neq 0$, that is, $f(b) = 0$ for some nonzero polynomial $f \in \Bbbk[x]$, the element b is *algebraic*. In this case, $\ker(\mathrm{ev}_b) = (\mu_b)$ is the principal ideal in $\Bbbk[x]$ generated by the minimal polynomial of b over $\Bbbk$, and $\Bbbk[b] = \Bbbk[x]/(\mu_b)$ has dimension $\deg \mu_b$ as a vector space over $\Bbbk$. This dimension is called the *degree* of b over $\Bbbk$ and denoted by $\deg_{\Bbbk}(b)$. Note that the algebraicity of b over $\Bbbk$ means the same as the integrality, and in this case, every element in $\Bbbk[b]$ is algebraic, and the algebra $\Bbbk[b]$ is a field if and only if it has no zero divisors. This certainly holds if B has no zero divisors. On the other side, $\Bbbk[b]$ has no zero divisors if and only if the minimal polynomial μ_b is irreducible in $\Bbbk[x]$.

6.5 Finitely generated algebras over a field. A commutative $\Bbbk$-algebra B with unit is said to be *finitely generated* if there are some elements $b_1, b_2, \dots, b_m \in B$ such that the evaluation map $\mathrm{ev}_{b_1,b_2,\dots,b_m} : \Bbbk[x_1, x_2, \dots, x_m] \to B$, $x_i \mapsto b_i$ for $i = 1, 2, \dots, m$, is surjective. In this case, $B = \Bbbk[x_1, x_2, \dots, x_m]/I$, where the ideal $I = \ker \mathrm{ev}_{b_1,b_2,\dots,b_m}$ consist of all *polynomial relations* between the *generators*[2] $b_1, b_2, \dots, b_m$ of the algebra B. It follows from the Corollary 6.1 and Exercise 6.3 on p. 71 that all finitely generated commutative $\Bbbk$-algebras are Noetherian, and the ideal of polynomial relations between any set of generators for such an algebra is finitely generated.

Theorem 6.2

If a finitely generated commutative $\Bbbk$-algebra B is a field, then every element of B is algebraic over $\Bbbk$.

Proof. Let elements $b_1, b_2, \dots, b_m$ generate B as an algebra over $\Bbbk$. We proceed by induction on m. The case $m = 1$, $B = \Bbbk[b]$, was already considered in n° 6.5. Let $m > 1$. If b_m is algebraic over $\Bbbk$, then $\Bbbk[b_m]$ is a field. By induction, B is algebraic over $\Bbbk[b_m]$, and Proposition 6.1 forces B to be algebraic over $\Bbbk$ as well. Thus, it is enough to check that b_m actually is algebraic over $\Bbbk$.

[1] That is, the monic polynomial $\mu_b \in Q_A[x]$ of minimal positive degree such that $\mu_b(b) = 0$.

[2] Generators of an algebra should be not confused with generators of a module. If elements $e_1, e_2, \dots, e_m$ span a ring B over a subring $A \subset B$ as a module, this means that B consists of finite A-liner combinations of these elements e_i, whereas if $b_1, b_2, \dots, b_m$ span B as an A-algebra, then B is formed by finite linear combinations of various monomials $b_1^{s_1} b_2^{s_2} \cdots b_m^{s_m}$.

Assume the contrary. Then the evaluation map (6-1) is injective for $b = b_m$, and is uniquely extended to an embedding of fields $\Bbbk(x) \hookrightarrow B$ by the universal property of the quotient field. Write $\Bbbk(b_m) \subset B$ for the image of this embedding. This is the smallest subfield in B containing b_m. By induction, B is algebraic over $\Bbbk(b_m)$. Therefore, every generator b_i, $1 \leqslant i \leqslant m-1$, is a root of some polynomial with coefficients in $\Bbbk(b_m)$. Multiplying this polynomial by an appropriate polynomial in b_m allows us to assume that all $(m-1)$ polynomials annihilating the generators $b_1, b_2, \dots, b_{m-1}$ have coefficients in $\Bbbk[b_m]$ and share the same leading coefficient, which we denote by $p(b_m) \in \Bbbk[b_m]$. Thus, the field B is integral over the subalgebra $F = \Bbbk[b_m, 1/p(b_m)] \subset B$ spanned over $\Bbbk$ by the elements b_m and $1/p(b_m)$. By the Proposition 6.3, F is a field. This forces p to be of positive degree, because otherwise $F = \Bbbk[b_m]$ is not a field. Now we claim that the element $1+p(b_m)$ has no inverse in F. Indeed, in the contrary case, there exists a polynomial $g \in \Bbbk[x_1, x_2]$ such that $g(b_m, 1/p(b_m)) \cdot (1 + p(b_m)) = 1$. Write the rational function $g(x, 1/p(x))$ as $h(x)/p^k(x)$, where $h \in \Bbbk[x]$ is not divisible by p in $\Bbbk[x]$. Then we get the polynomial relation $h(b_m) \cdot (p(b_m)+1) = p^k(b_m)$ on b_m. It is nontrivial, because the left hand side has positive degree and is not divisible by $p(x)$ in $\Bbbk[x]$. Contradiction. □

Corollary 6.5

Let a field $\mathbb{F}$ be finitely generated as an algebra over a subfield $\Bbbk \subset \mathbb{F}$. Then $\mathbb{F}$ has finite dimension as a vector space over $\Bbbk$.

Proof. If $\mathbb{F}$ is generated as a $\Bbbk$-algebra by algebraic elements $b_1, b_2, \dots, b_m$, then the monomials $b_1^{s_1} b_2^{s_2} \cdots b_m^{s_m}$ with $0 \leqslant s_i < \deg_{\Bbbk} b_i$ linearly span $\mathbb{F}$ over $\Bbbk$. □

6.6 Transcendence generators. Everywhere in this section we write A for a finitely generated $\Bbbk$-algebra without zero divisors, and Q_A for its field of fractions. Given a collection of elements $a_1, a_2, \dots, a_m \in A$, we write $\Bbbk(a_1, a_2, \dots, a_m) \subset Q_A$ for the smallest subfield containing all these elements.

Elements $a_1, a_2, \dots, a_m \in A$ are called *algebraically independent* if the evaluation map

$$\mathrm{ev}_{(a_1, a_2, \dots, a_m)} : \Bbbk[x_1, x_2, \dots, x_m] \to A,\ x_i \mapsto a_i,\ 1 \leqslant i \leqslant m,$$

is injective, that is, there are no polynomial relations between $a_1, a_2, \dots, a_m$. In this case the evaluation map is uniquely extended to the isomorphism of fields

$$\Bbbk(x_1, x_2, \dots, x_m) \xrightarrow{\sim} \Bbbk(a_1, a_2, \dots, a_m) \subset Q_A,$$

which maps a rational function of $(x_1, x_2, \dots, x_m)$ to its value at $(a_1, a_2, \dots, a_m)$.

Elements $a_1, a_2, \dots, a_m \in A$ are called *transcendence generators* of A over $\Bbbk$, if any element of A is algebraic over $\Bbbk(a_1, a_2, \dots, a_m)$. In this case the whole field Q_A is also algebraic over $\Bbbk(a_1, a_2, \dots, a_m)$, because the integer closure of $\Bbbk(a_1, a_2, \dots, a_m)$ in Q_A is a field by Proposition 6.3, and Q_A is contained in any field containing A by the universal property of the field of fractions.

An algebraically independent collection $a_1, a_2, \dots, a_m$ of transcendence generators of A over $\Bbbk$ is called a *transcendence basis* of A over $\Bbbk$. Since any proper subset of a transcendence basis is algebraically independent, the transcendence bases can be equivalently characterized as the minimal with respect to inclusions collections of transcendence generators, or as the maximal algebraically independent collections.

Similarly to the bases of vector spaces, any two transcendence bases of A have the same cardinality, and the proof is based on the same Exchange Lemma.

LEMMA 6.3 (EXCHANGE LEMMA)
Let elements $a_1, a_2, \dots, a_m$ be transcendence generators of A over $\Bbbk$, and let $b_1, b_2, \dots, b_n \in A$ be algebraically independent over $\Bbbk$. Then $n \leqslant m$, and after appropriate renumbering of the a_i and replacing the first n of them by $b_1, b_2, \dots, b_n$, the resulting elements $b_1, b_2, \dots, b_n, a_{n+1}, \dots, a_m$ are transcendence generators of A as well.

PROOF. Since b_1 is algebraic over $\Bbbk(a_1, a_2, \dots, a_m)$, there is a polynomial relation

$$f(b_1, a_1, a_2, \dots, a_m) = 0\,, \quad f \in \Bbbk[x_1, x_2, \dots, x_{m+1}]\,.$$

Since b_1 is transcendental over $\Bbbk$, this relation contains some a_i. After appropriate renumbering, we can assume that $i = 1$. Then a_1 and therefore all of Q_A is algebraic over $\Bbbk(b_1, a_2, \dots, a_m)$. Assume by induction that $b_1, \dots, b_k, a_{k+1}, \dots, a_m$ are transcendence generators of A over $\Bbbk$ for $k < n$. Since b_{k+1} is algebraic over $\Bbbk\left(b_1, \dots, b_k, a_{k+1}, \dots, a_m\right)$, there is a polynomial relation

$$f\left(b_1, \dots, b_k, b_{k+1}, a_{k+1}, \dots, a_m\right) = 0\,, \quad f \in \Bbbk[x_1, x_2, \dots, x_{m+1}]\,.$$

It must contain some a_{k+i}, because of algebraic independence of $b_1, b_2, \dots, b_n$ over $\Bbbk$. Hence, $m > k$ and after renumbering of the remaining elements a_i, we can assume that a_{k+1} is algebraic over $\Bbbk\left(b_1, \dots, b_{k+1}, a_{k+2}, \dots, a_m\right)$. Therefore, all of the Q_A is algebraic over this field too. This completes the induction step. □

COROLLARY 6.6
Let A be a finitely generated commutative $\Bbbk$-algebra without zero divisors. Then all transcendence bases of A over $\Bbbk$ have the same cardinality, any system of transcendence generators of A over $\Bbbk$ contains some transcendence basis, and every algebraically independent collection of elements in A can be included in a transcendence basis. □

DEFINITION 6.3
The cardinality of a transcendence basis of a finitely generated commutative $\Bbbk$-algebra A without zero divisors is called the *transcendence degree* of A and denoted $\operatorname{tr}\deg_{\Bbbk} A$.

EXAMPLE 6.3
Let $A \subset \Bbbk(t)$ be a $\Bbbk$-subalgebra different from $\Bbbk$. Then $\operatorname{tr}\deg_{\Bbbk} A = 1$. Indeed, for every

$$\psi = f(t)/g(t) \in A \smallsetminus \Bbbk\,,$$

the element t satisfies the algebraic equation $\psi \cdot g(x) - f(x) = 0$ with the coefficients in $\Bbbk(\psi)$. This forces the whole of $\Bbbk(t)$ to be algebraic over $\Bbbk(\psi) \subset \mathbb{Q}_A$ and ψ to be transcendental over $\Bbbk$, because otherwise, t would be algebraic over $\Bbbk$. Thus, any $\psi \in A \smallsetminus \Bbbk$ is a transcendence basis for both A and $\Bbbk(t)$.

6.7 Systems of polynomial equations. Any system of polynomial equations

$$f_\nu(x_1, x_2, \dots, x_n) = 0\,, \quad f_\nu \in \Bbbk[x_1, x_2, \dots, x_n]\,, \tag{6-2}$$

can be extended to a system whose left hand sides form the ideal $J \subset \Bbbk[x_1, x_2, \dots, x_n]$ spanned by the polynomials f_ν from (6-2). The extended infinite system has the same set of solutions in the affine space $\mathbb{A}^n = \operatorname{Aff}(\Bbbk^n)$ as the original system, because the equalities $f_\nu = 0$ imply the equalities $\sum_\nu g_\nu f_\nu = 0$ for all $g_\nu \in \Bbbk[x_1, x_2, \dots, x_n]$. Since the polynomial ring is Noetherian, the system $f = 0$, $f \in J$, is equivalent to a finite subsystem consisting of equations whose left hand sides generate J. Moreover, by the Lemma 6.1 on p. 70, this finite set of generators can be chosen among the original polynomials f_ν from (6-2). Thus, every (even infinite) system of polynomial equations is always equivalent, on the one hand, to some finite subsystem, and on the other hand, to a system of equations $f = 0$, where f runs through some ideal in $\Bbbk[x_1, x_2, \dots, x_n]$.

Given an ideal $J \subset \Bbbk[x_1, x_2, \dots, x_n]$, its zero set $V(J) \stackrel{\text{def}}{=} \{a \in \mathbb{A}^n \mid f(a) = 0 \ \ \forall f \in J\}$ is called an *affine algebraic variety* determined by J. Note that $V(J)$ may be empty. This happens, for example, if $J = (1) = \Bbbk[x_1, x_2, \dots, x_n]$ contains the equation $1 = 0$.

Associated with an arbitrary subset $\Phi \subset \mathbb{A}^n$ is the ideal

$$I(\Phi) \stackrel{\text{def}}{=} \{ f \in \Bbbk[x_1, x_2, \dots, x_n] \mid f(p) = 0 \text{ for all } p \in \Phi \},$$

called the *ideal of* Φ. Its zero set $V(I(\Phi))$ is the smallest affine algebraic variety containing Φ. For every ideal $J \subset \Bbbk[x_1, x_2, \dots, x_n]$ there is the tautological inclusion $J \subset I(V(J))$. In general, it is proper. Say, for $n = 1$, the ideal $J = (x^2) \subset \Bbbk[x]$ determines the variety $V(x^2) = \{0\} \subset \mathbb{A}^1$ whose ideal is $I(V(x^2)) = (x) \supsetneq (x^2)$.

THEOREM 6.3 (HILBERT'S NULLSTELLENSATZ)
Let $\Bbbk$ be an algebraically closed field, $J \subset \Bbbk[x_1, x_2, \dots, x_n]$ an ideal, $\sqrt{J} \stackrel{\text{def}}{=} \{f \mid \exists\, m \in \mathbb{N} : f^m \in J\}$ the *radical* of J. Then $I(V(J)) = \sqrt{J}$ (the *strong Nullstellensatz*). In particular, $V(J) = \varnothing$ if and only if $1 \in J$ (the *week Nullstellensatz*).

PROOF. Let us prove the week Nullstellensatz first. It is enough to show that for any proper ideal $J \subset \Bbbk[x_1, x_2, \dots, x_n]$, there exists a point $p \in \mathbb{A}^n$ such that $f(p) = 0$ for all $f \in J$. Without loss of generality the ideal J can replaced by a *maximal* proper ideal $\mathfrak{m} \supset J$.

EXERCISE 6.6. Convince yourself that an ideal $\mathfrak{m}$ in a commutative ring K is maximal among the proper ideals of K partially ordered by inclusions if and only if the quotient ring $K/\mathfrak{m}$ is a field.

Thus, we can assume that the quotient ring $\Bbbk[x_1, x_2, \dots, x_n]/\mathfrak{m}$ is a field. Since it is finitely generated as a $\Bbbk$-algebra, the Theorem 6.2 forces every element $\vartheta \in \Bbbk[x_1, x_2, \dots, x_n]/\mathfrak{m}$ to be algebraic over $\Bbbk$, that is, to satisfy an equation $\mu(\vartheta) = 0$ for a monic irreducible polynomial $\mu \in \Bbbk[t]$. Since $\Bbbk$ is algebraically closed, the polynomial μ has to be linear, and therefore, $\vartheta \in \Bbbk$. In other words, every polynomial is congruent modulo $\mathfrak{m}$ to a constant. Write $p_i \in \Bbbk$ for the constant congruent to x_i. Then the factorization homomorphism $\Bbbk[x_1, x_2, \dots, x_n] \to \Bbbk[x_1, x_2, \dots, x_n]/\mathfrak{m} \simeq \Bbbk$ maps every polynomial $f(x_1, x_2, \dots, x_n)$ to the class of constant $f(p_1, p_2, \dots, p_n) \in \Bbbk$. Since all $f \in \mathfrak{m}$ are mapped to zero, they all vanish at $p = (p_1, p_2, \dots, p_n) \in \mathbb{A}^n$, as desired.

The strong Nullstellensatz is trivial for $V(J) = \varnothing$. Assume that $V(J) \neq \varnothing$, that is, $J \neq (1)$. Consider $\mathbb{A}^n$ as the hyperplane $t = 0$ in the affine space $\mathbb{A}^{n+1}$ with the coordinates

$$(t,\, x_1, x_2, \dots, x_n).$$

If a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n] \subset \Bbbk[t,\, x_1, x_2, \dots, x_n]$ vanishes everywhere on the cylinder $V(J) \subset \mathbb{A}^{n+1}$, then the polynomial $g(t, x) = 1 - t\,f(x)$ equals 1 at every point of $V(J)$. Therefore, the ideal spanned in $\Bbbk[t,\, x_1, x_2, \dots, x_n]$ by J and $g(t, x)$ has the empty zero set in $\mathbb{A}^{n+1}$. By the week Nullstellensatz, this ideal contains 1, i.e., there exist $q_0, q_1, \dots, q_s \in \Bbbk[t,\, x_1, x_2, \dots, x_n]$ and $f_1, f_2, \dots, f_s \in J$ such that $q_0(x, t) \cdot \big(1 - t f(x)\big) + q_1(t, x) \cdot f_1(x) + \dots + q_s(x, t) \cdot f_s(x) = 1$. The homomorphism $\Bbbk[t, x_1, x_2, \dots, x_n] \to \Bbbk(x_1, x_2, \dots, x_n)$ acting on the variables as $t \mapsto 1/f(x)$, $x_\nu \mapsto x_\nu$ for $1 \leqslant \nu \leqslant n$, maps this equality to the equality

$$q_1\big(1/f(x), x\big) \cdot f_1(x) + \cdots + q_s\big(1/f(x), x\big) \cdot f_s(x) = 1\,. \tag{6-3}$$

in the field $\Bbbk(x_1, x_2, \dots, x_n)$. Since $1 \notin J$, some $q_\nu\big(1/f(x), x\big)$ have nontrivial denominators. All these denominators are canceled via multiplication by f^m for some $m \in \mathbb{N}$. Multiplying both sides by this f^m leads to the required equality $f^m(x) = \widetilde{q}_1(x) \cdot f_1(x) + \cdots + \widetilde{q}_s(x) \cdot f_s(x)$ with $\widetilde{q}_\nu \in \Bbbk[x_1, x_2, \dots, x_n]$. □

6.8 Resultants. Given a system of homogeneous polynomial equations

$$\begin{cases} f_1(x_0, x_1, \dots, x_n) = 0 \\ f_2(x_0, x_1, \dots, x_n) = 0 \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_0, x_1, \dots, x_n) = 0\,, \end{cases} \tag{6-4}$$

where every $f_i \in \Bbbk[x_0, x_1, \dots, x_n]$ is homogeneous of degree d_i, the set of its solutions, considered up to proportionality, is the intersection of m projective hypersurfaces $S_i = V(f_i) \subset \mathbb{P}(V)$, where $V = \Bbbk^{n+1}$. The projective hypersurfaces of degree d in $\mathbb{P}(V)$ can be viewed as points of the projective space $\mathbb{P}(S^d V^*)$. All collections of hypersurfaces $(S_1, S_2, \dots, S_m)$ of given degrees $d_1, d_2, \dots, d_m$ with nonempty intersection $\bigcap_i S_i \neq \varnothing$ form the figure

$$\mathcal{R}(n+1;\, d_1, d_2, \dots, d_m) \subset \mathbb{P}(S^{d_1}V^*) \times \mathbb{P}(S^{d_2}V^*) \times \cdots \times \mathbb{P}(S^{d_m}V^*)\,, \tag{6-5}$$

called the *resultant variety* of the homogeneous system (6-4). When $m = n+1$ and all $d_i = 1$, the system (6-4) becomes the system of linear equations $Ax = 0$ with the square matrix $A = \big(a_{ij}\big)$. It has a nonzero solution if and only if $\det\big(a_{ij}\big) = 0$. Thus, in this simplest case, the resultant variety is a projective variety determined by one multilinear equation of total degree $n+1$ on the coefficients $a_{i,j}$. We are going to check that the resultant variety (6-5) can always be described by a system of polynomial equations in the coefficients of the polynomials f_i. This system is called a *resultant system*. It depends only on the number of variables and the collection of degrees $d_1, d_2, \dots, d_m$. Every resultant equation is homogeneous in the coefficients of each polynomial.

Write $J = (f_1, f_2, \dots, f_m) \subset \Bbbk[x_0, x_1, \dots, x_n]$ for the ideal spanned by the polynomials (6-4). If $V(J)$ is exhausted by the origin, then every coordinate linear form x_i vanishes on $V(J)$, and therefore, all $x_i^m \in J$ for some $m \in \mathbb{N}$ by the strong Nullstellensatz. This forces J to contain all homogeneous polynomials of degree $d > (m-1)(n+1)$. Conversely, if $J \supset S^d V^*$ for all $d \gg 0$, then the system (6-4) implies the equations $x_0^d = x_1^d = \cdots = x_n^d = 0$, and therefore, has only the zero solution. For any $d \in \mathbb{N}$, the intersection $J \cap S^d V^*$ coincides with the image of $\Bbbk$-linear map

$$\mu_d : S^{d-d_1}V^* \oplus S^{d-d_2}V^* \oplus \cdots \oplus S^{d-d_m}V^* \xrightarrow{(g_0, g_1, \dots, g_n) \mapsto \sum g_\nu f_\nu} S^d\,. \tag{6-6}$$

The matrix of this map in the standard monomial bases consists of zeros and the coefficients of polynomials f_ν. For $d \gg 0$, the dimension of the left hand side in (6-6) grows as

$$\sum_{\nu=1}^{m} \binom{n+d-d_\nu}{n} \sim \frac{m}{n!}\, d^n$$

and becomes greater that the dimension of the right hand side, which grows as

$$\binom{n+d}{n} \sim \frac{1}{n!}\, d^n .$$

Thus, for every $d \gg 0$, the condition $S^d V^* \not\subset J$, that is, the non-surjectivity of the map (6-6), means that the rank of the matrix of μ_d is not maximal. This is equivalent to the vanishing of all minors of the maximal degree in the matrix. Thus, the resultant variety is the zero set of all these minors written for all d such that the dimension of the left hand side of (6-6) is not less than that of the right han side. Since the polynomial ring is Noetherian, this huge system of equations is equivalent to some finite subsystem. If the ideal of the resultant variety (6-5) is not principal, such a system of resultants is not unique in general.

Example 6.4 (resultant of two binary forms)
Let the ground field $\Bbbk$ be algebraically closed. Then every homogeneous binary form of degree d

$$f(t_0,t_1) = a_0\, t_1^d + a_1\, t_0\, t_1^{d-1} + a_2\, t_0^2\, t_1^{d-2} + \cdots + a_{d-1}\, t_0^{d-1}\, t_1 + a_d\, t_0^d$$

has d roots $\alpha_1, \alpha_2, \dots, \alpha_d$, $\alpha_i = (\alpha_i' : \alpha_i'')$, on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ and is factorized as

$$f(t_0,t_1) = \prod_{i=0}^{d} (\alpha_i'' t_0 - \alpha_i' t_1) = \prod_{i=0}^{d} \det \begin{pmatrix} t_0 & t_1 \\ \alpha_i' & \alpha_i'' \end{pmatrix}$$

The coefficients of f are expressed as the homogeneous polynomials in the roots by means of the *homogeneous Viète's formulas*: $a_k = (-1)^{d-k} \sigma_k(\alpha', \alpha'')$, where

$$\sigma_k(\alpha', \alpha'') = \sum_{\#I=k} \Bigl(\prod_{i\in I} \alpha_i' \cdot \prod_{j\notin I} \alpha_j'' \Bigr)$$

and I runs through the strictly increasing sequences of k indexes. In particular, a_k is bihomogeneous of bidegree $(k, d-k)$ in (α', α''). Let us fix two degrees $r, s \in \mathbb{N}$ and consider the polynomial ring $\Bbbk[\alpha', \alpha'', \beta', \beta'']$ in four collections of variables

$$\begin{aligned} \alpha' &= (\alpha_1', \alpha_2', \dots, \alpha_s') & \alpha'' &= (\alpha_1'', \alpha_2'', \dots, \alpha_s'') \\ \beta' &= (\beta_1', \beta_2', \dots, \beta_r') & \beta'' &= (\beta_1'', \beta_2'', \dots, \beta_r'') . \end{aligned}$$

Within this ring, consider the product

$$R \stackrel{\text{def}}{=} \prod_{i,j} \Bigl(\alpha_i' \beta_j'' - \alpha_i'' \beta_j' \Bigr) = \prod_{j=1}^{s} f(\beta_j) = (-1)^{rs} \prod_{i=1}^{r} g(\alpha_i) .$$

The polynomial R is bihomogeneous of bidigree (rs, rs) in (α, β). It is evaluated to zero at the roots α, β of binary forms $f(t_0,t_1) = \sum_{i=0}^{s} a_i\, t_0^i\, t_1^{n-i}$, $g(t_0,t_1) = \sum_{j=0}^{r} b_j\, t_0^j\, t_1^{m-j}$ if and only if these forms have a common root in $\mathbb{P}_1$. Let us show that R is expressed as a polynomial R_{fg} in the coefficients $a_i = (-1)^{n-i}\sigma_i(\alpha', \alpha'')$, $b_j = (-1)^{m-j}\sigma_j(\beta', \beta'')$ of f, g by the following *Sylvester* *formula*

$$R_{fg} = \det \underbrace{\begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_s & & & \\ & a_0 & a_1 & \cdots & \cdots & a_s & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & a_s \\ b_0 & b_1 & \cdots & \cdots & b_r & & & \\ & b_0 & b_1 & \cdots & \cdots & b_r & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & b_0 & b_1 & \cdots & \cdots & b_r \end{pmatrix}}_{r+s} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} r \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} s \end{matrix} \tag{6-7}$$

where the matrix in the right hand side is transposed to the matrix of the linear map (6-6) written for $n = 1$, $m = 2$, and $d = s + r - 1$ in the standard monomial bases, when it turns to

$$\mu_{s+r-1} : S^{r-1}V^* \oplus S^{s-1}V^* \to S^{s+r-1}V^*, \quad (h_1, h_2) \mapsto fh_1 + gh_2, \quad \dim V = 2,$$

and the both sides are of equal dimension $r + s$.

Exercise 6.7. Verify this carefully.

Write $S = S(\alpha', \alpha'', \beta', \beta'') \in \Bbbk[\alpha', \alpha'', \beta', \beta'']$ for the Sylvester determinant from the right hand side of (6-7), and put $D_{ij} = \alpha_i'\beta_j'' - \alpha_i''\beta_j'$. For every point $(\alpha, \beta) \in V(D_{ij})$, we have the equality $(\alpha_i'' t_0 - \alpha_i' t_1) = (\beta_i'' t_0 - \beta_i' t_1)$ up to a constant factor, and this linear form divides $f(t)$, $g(t)$, and all polynomials $f(t)h_1(t) + g(t)h_2(t)$ in $\Bbbk[t_0, t_1]$. Hence, $\operatorname{im} \mu_{r+s-1} \neq S^{r+s-1}V^*$, and therefore, $S(\alpha, \beta) = 0$. Thus, S vanishes identically on $V(D_{ij})$. By the strong Nullstellensatz, some power of S is divisible by D_{ij}. Since D_{ij} is irreducible and the polynomial ring $\Bbbk[\alpha', \alpha'', \beta', \beta'']$ is factorial, D_{ij} divides S, and therefore, S is divisible by the product R_{fg} of all D_{ij}. Comparison of the degrees and coefficients of the lexicographically maximal monomials in S and R_{fg} shows that these two polynomials must be equal.

We conclude that the resultant variety (6-5) for a pair of binary forms f, g of degrees s, r is the hypersurface[1] in $\mathbb{P}_s \times \mathbb{P}_r$ determined by one equation $R_{fg} = 0$ on the coefficients of f, g. The polynomial $R_{f,g}$ is called the *resultant* of f, g. For $t_0 = 1$, $t_1 = x$, it is specialized to the resultant $R_{f_{\text{aff}}, g_{\text{aff}}}$ of two non-homogeneous polynomials $f_{\text{aff}}(x) = f(1, x)$, $g_{\text{aff}}(x) = g(1, x)$ in one variable x. Under the assumption that[2] $a_0 b_0 \neq 0$, the resultant $R_{f_{\text{aff}}, g_{\text{aff}}}$ vanishes if and only if the polynomials f_{aff}, g_{aff} have a common root in $\Bbbk$.

[1] In **??** on p. **??**, we will see that the same holds for any system of homogeneous polynomial equations in which the number of equations equals the number of unknowns.

[2] It says that both binary forms f, g do not vanish at the point $(0 : 1)$, the infinity of the affine chart U_0 in which the coordinate x is defined.

Comments to some exercises

EXRC. 1.4. The right hand side consists of $q^n + q^{n-1} + \cdots + q + 1$ points. The cardinality of the left hand side equals the number of non zero vectors in $\mathbb{F}_q^{n+1}$ divided by the number of non zero elements in $\mathbb{F}_q$, that is, $(q^{n+1}-1)/(q-1)$. We get the summation formula for geometric progression.

EXRC. 1.5. Every line passing through the origin of $\mathbb{R}^{n+1}$ intersects the unit semisphere $\sum x_i^2 = 1$, $x_0 \geqslant 0$. The lines laying in the hyperplane $x_0 = 0$ intersect the semisphere in two opposite points of the boundary. Any other line intersects the semisphere in exactly one internal point. Thus, $\mathbb{P}(\mathbb{R}^{n+1})$ can be obtained from the solid ball of dimension n by gluing together every pair of opposite points of its boundary sphere. In particular, the plane $\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3)$ is obtained from a square by gluing the opposite edges taken with opposite orientations, see fig. 6⋄1.

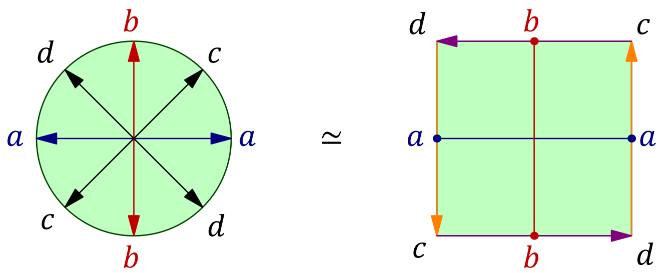


Fig. 6⋄1. Gluing $\mathbb{P}(\mathbb{R}^3)$ from a square.

The same result is obtained by gluing a Möbius tape with a disk along the boundary circles, see fig. 6⋄2.

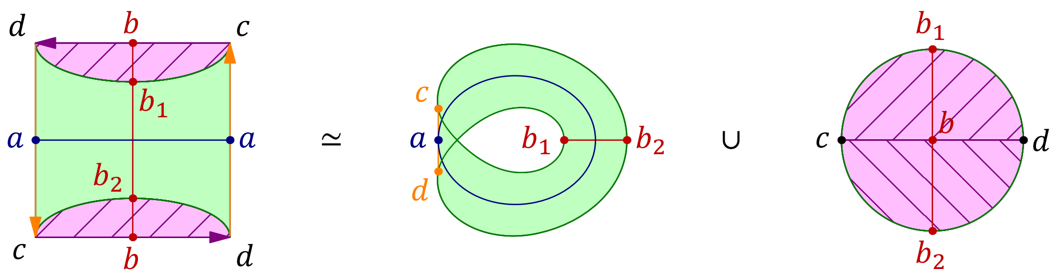


Fig. 6⋄2. $\mathbb{P}(\mathbb{R}^3)$ as a Möbius tape glued to a disk along the boundary circle.

The solid ball of radius π in $\mathbb{R}^3$ is mapped onto the group SO_3 by sending a point P to the rotation about line (OP) by angle[1] $|OP|$ radians in the clockwise direction being viewed along $\overrightarrow{OP}$. This map is injective on internal points of the ball and identifies the opposite points of its boundary sphere.

EXRC. 1.6. Let $\mathbb{P}_n = \mathbb{P}(V)$, $K = \mathbb{P}(U)$, $L = \mathbb{P}(W)$ for some vector subspaces $U, W \subset V$. Then

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) \geqslant \dim(K) + 1 + \dim(L) + 1 - n - 1 \geqslant 1\,.$$

EXRC. 1.7. $\binom{n+d}{d} - 1$.

EXRC. 1.8. In projective space any line does intersect any hyperplane, see Exercise 1.6.

[1] We write $|AB|$ for the euclidean distance between the points A, B.

Exrc. 1.9. If char $\Bbbk = p > 0$ and $d = pm$, then $(\alpha_0 x_0 + \alpha_1 x_1)^d = (\alpha_0^p x_0^p + \alpha_1^p x_1^p)^m$ lies in the linear span of those monomials $x_0^\mu x_1^\nu$ whose exponents μ, ν both are divisible by p.

Exrc. 1.10. Let vector $v = u + w$ represent a point $p \in \mathbb{P}(V)$. Then $\ell = (u, w)$ passes through p and intersects K and L at u and w. Vice versa, if $v \in (a, b)$, where $a \in U$ and $b \in W$, then $v = \alpha a + \beta b$ and the uniqueness of the decomposition $v = u + w$ forces $\alpha a = u$ and $\beta b = w$. Hence $(ab) = \ell$.

Exrc. 1.12. Let $L_1 = \mathbb{P}(U)$, $L_2 = \mathbb{P}(W)$, $p = \mathbb{P}(\Bbbk \cdot e)$. Then $V = W \oplus \Bbbk \cdot e$, because of $p \notin L_2$. Projection from p is a projectivization of linear projection of V onto W along $\Bbbk \cdot e$. Since $p \notin L_1$, the restriction of this projection onto U has zero kernel. Thus, it produces linear projective isomorphism.

Exrc. 1.13. Let $[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$. Write $\varphi_p, \varphi_q : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ for the linear projective automorphisms sending ∞, 0, 1 to the triples p_1, p_2, p_3 and q_1, q_2, q_3 respectively. Then $\varphi_p(p_4) = \varphi_q(q_4)$ and $\varphi_q^{-1} \circ \varphi_p$ sends p_1, p_2, p_3, p_4 to q_1, q_2, q_3, q_4. Vice versa, let a linear projective automorphism $\psi_{pq} : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ send q_1, q_2, q_3, q_4 to p_1, p_2, p_3, p_4. Write $\psi_p : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ for the linear projective automorphism sending p_1, p_2, p_3 to ∞, 0, 1. Then $\psi_p \circ \psi_{pq}$ takes

$$q_1, q_2, q_3, q_4 \mapsto \infty, 0, 1, [p_1, p_2, p_3, p_4].$$

Hence, $[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$.

Exrc. 1.14. The map $(p_2, p_1, p_3) \mapsto (\infty, 0, 1)$ can be decomposed as the map $(p_1, p_2, p_3) \mapsto (\infty, 0, 1)$ followed by the map $(\infty, 0, 1) \mapsto (0, \infty, 1)$, which takes $\vartheta \mapsto 1/\vartheta$. Similarly, to permute (p_1, p_2, p_3) via the cycles (13), (23), (123), (132) we compose the map $(p_1, p_2, p_3) \mapsto (\infty, 0, 1)$ with the maps sending $(\infty, 0, 1)$ to $(1, 0, \infty)$, $(\infty, 1, 0)$, $(1, \infty, 0)$, $(0, 1, \infty)$ respectively, i.e., with the maps sending ϑ to $\vartheta/(\vartheta - 1)$, $1 - \vartheta$, $(\vartheta - 1)/\vartheta$, $1/(1 - \vartheta)$.

Exrc. 2.4. This follows from the last representation from formula (2-1) on p. 16.

Exrc. 2.5. Let $\mathbb{P}(V) = \mathbb{P}(\operatorname{Ann} \xi) \cup \mathbb{P}(\operatorname{Ann} \eta)$ for some non zero covectors $\xi, \eta \in V^*$. Then the quadratic form $q(v) = \xi(v)\eta(v)$ vanishes identically on V. Therefore its polarization $\widetilde{q}(u, w) = (q(u + w) - q(u) - q(w))/2$ also vanishes. Hence, the Gram matrix of q equals zero, i.e., q is the zero polynomial. However, the polynomial ring has no zero divisors.

Exrc. 2.7. Use Lemma 2.1 on p. 18 and prove that non-empty smooth quadric over an infinite field can not be covered by a finite number of hyperplanes.

Exrc. 2.9. Pick up some 3 on each line and draw a quadric through these 9 points.

Exrc. 2.10. By Theorem 2.1 on p. 18, S is the linear join of the singular line $\operatorname{Sing} S$ and a smooth quadric $S \cap \ell$ within a line ℓ complementary to $\operatorname{Sing} S$. This smooth quadric is either a pair of distinct points or empty.

Exrc. 2.12. Every line $\ell \subset S$ passing through a given point $a \in S$ lies inside $S \cap T_a S$, which is the split conic exhausted by two ruling lines crossing at a.

Exrc. 2.13. See Proposition 2.10 on p. 23.

Exrc. 2.14. Use *the method of loci*: remove one of the given lines and look how does the locus filled by the lines crossing 3 remaining lines interact with the removed line.

Exrc. 3.1. This is a particular case of Exercise 1.12.

Exrc. 3.2. Draw the cross-axix ℓ by joining $(a_1 b_2) \cap (b_1, a_2)$ and $(c_1 b_2) \cap (b_1, c_2))$. Then draw a line through b_1 and $\ell \cap (x, b_2)$. This line crosses ℓ_2 in $\varphi(x)$.

EXRC. 3.3. Let two tangent lines to C drown from x be given by linear equations $\xi(x) = 0$, $\eta(x) = 0$, and let the line ℓ_1 be the second of them. Then $\xi, \eta \in \mathbb{P}_2^\times$ are the intersection points of the dual conic $C^\times \subset \mathbb{P}_2$ and the line $\operatorname{Ann} x \subset \mathbb{P}_2^\times$. To find them, we need to solve a quadratic equation whose coefficients are polynomials in the coordinates of the point x and the elements of the Gram matrix of conic C. One root of this equation leads to the given point $\eta \in \mathbb{P}_2$ and therefore is known. Then the second root is a rational function of the first root and the coefficients of quadratic equation by the Vieta formula.

EXRC. 3.4. The arguments are dual to those from Exercise 3.3.

EXRC. 3.6. Let $c_1, c_2 \in C \smallsetminus \{a_1, a_2, a_3, a_4\}$. Parametrize the pencils $c_1^\times$ and $c_2^\times$ by some lines $\ell_1 \not\ni c_1$ and $\ell_2 \not\ni c_2$ respectively, and write a_i', a_i'' for the images of points a_i under the projections $c_i : D \xrightarrow{\sim} \ell_i$. Then $[a_1, a_2, a_3, a_4] = [a_1', a_2', a_3', a_4'] = [a_1'', a_2'', a_3'', a_4'']$, where the second equality holds, because the composition of projections $(c_2 : D \xrightarrow{\sim} \ell_2) \circ (c_1 : \ell_1 \xrightarrow{\sim} D)$ is a homography $\ell_1 \xrightarrow{\sim} \ell_2$ sending $a_i \mapsto a_i''$ for all i (comp. with n° 3.1.3 on p. 28). Since any linear projective automorphism $\varphi : \mathbb{P}_2 \xrightarrow{\sim} \mathbb{P}_2$ induces the homography of the pencils of lines $a^\times \xrightarrow{\sim} \varphi(a)^\times$, the second statement of the problem holds as well.

EXRC. 3.8. This is the smooth conic passing through p, q, a, b, c.

EXRC. 3.10. For given $p, q \in \mathbb{P}_1$, the equality $[p, q, x, y] = -1$ allows to express $x = x_0 / x_1$ and $y = y_0 / y_1$ through one other rationally. Hence, by Lemma 3.1 on p. 26, a homography $\mathbb{P}_1 \to \mathbb{P}_1$ is provided by the map sending a point $x \in \mathbb{P}_1$ to the point $y \in \mathbb{P}_1$ such that $[p, q, x, y] = -1$. It is involutive[1], because $[p, q, x, y] = -1 = [p, q, y, x]$. Since it keeps both p, q fixed, it coincides with $\sigma_{p,q}$.

EXRC. 3.13. For a point p and line ℓ in $\mathbb{P}_2 = \mathbb{P}(V)$, the conics $C = V(f) \subset \mathbb{P}_2$ such that ℓ is the polar of p with respect to C form a projective subspace of codimension 2 in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$. Indeed, associated with $p \in V$ is the linear map

$$\operatorname{pl}_p : S^2V^* \to V^*, \quad q \mapsto \hat{q}(p), \tag{6-8}$$

which sends a quadratic form q to the covector $\hat{q}(p) : V \to \Bbbk$, and $\dim \ker \operatorname{pl}_v = \dim S^2V^* - \dim V^* = 3$ when $\dim V = 3$. Thus, the preimage of dimension 1 subspace $\operatorname{Ann}(\ell) \in V^*$ under the map (6-8) has dimension 4, that is, codimension 2. Its projectivisation is of codimension 2 as well. In particular, for $p \in \ell$, this gives what we have stated. Futher, two subspaces of codimension 2 in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$ formed, respectively, by conics touching the lines ℓ_1, ℓ_2 at the points $p_1 \in \ell_1 \smallsetminus \ell_2$, $p_2 \in \ell_2 \smallsetminus \ell_1$ are intersecting at least along a line. If their intersection would a plane, then for any pair of points $a, b\mathbb{P}_2$ there would be a conic passing through a, b and touching ℓ_1, ℓ_2 at p_1, p_2 respectively. For $a \in \ell \smallsetminus \{p_1, p_2\}$, $b \notin \ell \cup \ell_1 \cup \ell_2$, such the conic must split into the line ℓ and another line different from ℓ, ℓ_1, ℓ_2. Hence, this conic can not intersect ℓ_1, ℓ_2 with multiplicities 2 in p_1, p_2 simultaneously.

EXRC. 3.14. The first follows from the fact that $\ell_1'' \cup \ell_2''$ also touches ℓ at p_1. The second is similar to Exercise 3.13: use the facts that conics passing through a given point form a hyperplane, whereas conics touching a given line at a given point form a subspace of codimension 2 in the space of conics.

EXRC. 3.15. Four hyperplanes in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$ formed by the conics passing through a, b, c, d are linearly independent, because for any 3 of the points, there is a split conic passing through them

[1] Do you see that in the affine chart whose infinity is p, the this homography is nothing but the central symmetry with respect to q?

but not through the remaining fourth point. Hence, these 4 hyperplanes are intersecting along a line. The split conics formed by pairs of opposite sides in quadrangle $abcd$ lie in the pencil. This forces the pencil to be simple.

EXRC. 4.3. The first statement is verified by the same arguments as in **??** on p. **??** and n° 2.5.1. To prove the second, chose some dual bases $u_1, u_2, \dots, u_n \in U$, $u_1^*, u_2^*, \dots, u_n^* \in U^*$ and a basis $w_1, w_2, \dots, w_m \in W$. Then mn decomposable tensors $u_i^* \otimes w_j$ form a basis in $U^* \otimes V$. The matrix of operator

$$u_i^* \otimes w_j : u_k \mapsto \begin{cases} w_j & \text{for } k = i \\ 0 & \text{otherwise} \end{cases}.$$

has 1 in the crossing of j th row with i th column and zeros elsewhere. Thus, these operators span $\operatorname{Hom}(U, W)$.

EXRC. 4.4. For any linear mapping $f : V \to A$ the multiplication

$$V \times V \times \cdots \times V \to A,$$

which takes $(v_1, v_2, \dots, v_n)$ to their product $\varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n) \in A$, is multilinear. Hence, for each $n \in \mathbb{N}$ there exists a unique linear mapping $V^{\otimes n} \to A$ taking tensor multiplication to multiplication in A. Add them all together and get required algebra homomorphism $\mathsf{T}V \to A$ extending f. Since any algebra homomorphism $\mathsf{T}V \to A$ that extends f has to take $v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n)$, it coincides with the extension just constructed. Uniqueness of free algebra is proved exactly like Lemma 4.1 on p. 39.

EXRC. 4.5. Since the decomposable tensors span $V^{*\otimes n}$ and the equality

$$i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v, w_1, w_2, \dots, w_{n-1})$$

is bilinear in v, φ, it is enough to check it for the decomposable $\varphi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$.

EXRC. 4.6. Fix a basis $e_1, \dots, e_p, u_1, \dots, u_q, w_1, \dots, w_r, v_1, \dots, v_s$ in V such that e_i form a basis in $U \cap W$, u_j and w_k extend it to some bases in U, W, and v_m complete everything to a basis in V. Then expand t through the standard monomial basis of $\mathsf{T}V$ built from this basis of V.

EXRC. 4.8. Fo all $v, w \in V$ we have

$$0 = \varphi(\dots, (v + w), \dots, (v + w), \dots) = \varphi(\dots, v, \dots, w, \dots) + \varphi(\dots, w, \dots, v, \dots).$$

Vice versa, if $\operatorname{char} \Bbbk \neq 2$, then $\varphi(\dots, v, \dots, v, \dots) = -\varphi(\dots, v, \dots, v, \dots)$ forces

$$\varphi(\dots, v, \dots, v, \dots) = 0.$$

EXRC. 4.9. See, e.g., the Proposition 11.2 on p. 260 in the sec. 11.2.2 of the book: A. L. Gorodentsev, *Algebra I. Textbook for Students of Mathematics.*, Springer, 2016.

EXRC. 4.10. Every multilinear map $\varphi : V \times V \times \cdots \times V \to W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $F : V^{\otimes n} \to W$ is linear. Such F is factorized through the projection $V^{\otimes n} \twoheadrightarrow S^nV$ if and only if

$$F(\cdots \otimes v \otimes w \otimes \cdots) = F(\cdots \otimes w \otimes v \otimes \cdots).$$

The latter is equivalent to $\varphi(\dots, v, w, \dots) = \varphi(\dots, w, v, \dots)$. This proves the universality of the multiplication in SV. Every linear map $f : V \to A$ induces the symmetric multilinear map

$V \times V \times \cdots \times V \to A$, $(v_1, v_2, \dots, v_n) \mapsto \prod \varphi(v_i)$ for any $n \in \mathbb{N}$. The latter gives the linear map $S^n V \to A$. All together these maps extend f to the homomorphism of $\Bbbk$-algebras $SV \to A$. Vice versa, every homomorphism of $\Bbbk$-algebras $SV \to A$, which extends f, takes $\prod v_i \to \prod \varphi(v_i)$ and coincides with the previous extension. The uniqueness of extension is verified as in Lemma 4.1 on p. 39.

EXRC. 4.11. The first follows from $0 = (v + w) \otimes (v + w) = v \otimes w + w \otimes v$, the second from $v \otimes v + v \otimes v = 0$.

EXRC. 4.12. Similar to **??** on p. **??**.

EXRC. 4.13. If $\dim V = d$, then $Z(\Lambda V) = \Lambda^d V + \bigoplus_k \Lambda^{2k} V$. For even d, the first summand is contained in the second, for odd d the sum is direct.

EXRC. 4.15. Use that $\det A = \det A^t$, and transpose everything.

EXRC. 4.16. The summands form one S_n-orbit. The stabilizer of an element in this orbit consists of $m_1!\, m_2! \cdots m_d!$ independent permutations of coinciding factors. Hence, the length of orbit equals $\frac{n!}{m_1!\, m_2! \cdots m_d!}$.

EXRC. 4.17. For $v = \sum \alpha_i e_i$, the complete contraction of $v^{\otimes n}$ with $\widetilde{f} = \frac{m_1! \cdot m_2! \cdots m_d!}{n!} x_{[m_1, m_2, \dots, m_d]}$ is the sum of $n!/(m_1! \cdot m_2! \cdots m_d!)$ mutually equal products

$$\frac{m_1! \cdot m_2! \cdots m_d!}{n!} \cdot x_1(v)^{m_1} \cdot x_2(v)^{m_2} \cdot \cdots \cdot x_d(v)^{m_d} = \frac{m_1! \cdot m_2! \cdots m_d!}{n!} \cdot \alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_d^{m_d}\,.$$

Thus, it coincides with the result of substitution $(x_1, x_2, \dots, x_n) = (\alpha_1, \alpha_2, \dots, \alpha_n)$ in the monomial $\frac{n!}{m_1! m_2! \cdots m_d!} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$.

EXRC. 4.18. Use the same arguments as in the proof of multinomial expansion formula

$$(v_1 + v_2 + \cdots + v_k)^n = \sum_{m_1 m_2 \dots m_k} \frac{n!}{m_1!\, m_2! \cdots m_k!} \cdot v_1^{m_1} v_2^{m_2} \cdots v_k^{m_k}\,.$$

EXRC. 4.20. Since the Leibniz rule is linear in v, f, g, it is enough to check it for $v = e_i$, $f = x_1^{m_1} \dots x_d^{m_d}$, $g = x_1^{k_1} \dots x_d^{k_d}$. In this case it follows directly from the definition of polar map. The formula for $\widetilde{f}(v_1, v_2, \dots, v_n)$ follows from the equality $\widetilde{f}(v_1, x, \dots, x) = \frac{1}{n} \cdot \partial_{v_1} f(x)$ by induction in $n = \deg f$.

EXRC. 4.23. Similar to Exercise 4.20.

EXRC. 4.24. Let $e_1, e_2, \dots, e_m$ be a basis in U. If $\omega \notin \Lambda^m U$, then the expansion of ω as a linear combination of basis monomials e_I contains a monomial whose index I differs from the whole $1, 2, \dots, m$. Let $k \notin I$. Then $e_k \wedge \omega \neq 0$, because the basis monomial $e_{\{k\} \sqcup I}$ appears in $e_k \wedge \omega$ with a nonzero coefficient. Conversely, if $\omega \in \Lambda^m U$, then $\omega = \lambda \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_m$ and $e_i \wedge \omega = 0$ for all i.

EXRC. 4.26. See Example 4.9 on p. 57.

EXRC. 4.27. Let $U \neq W$ be two subspaces of dimension m. Chose a basis

$$e_1, e_2, \dots, e_r,\ u_1, u_2, \dots, u_{m-r},\ w_1, w_2, \dots, w_{m-r},\ v_1, v_2, \dots, v_{d+r-2m} \in V$$

such that $e_1, e_2, \dots, e_r$ is a basis of $U \cap W$, vectors $u_1, u_2, \dots, u_{m-r}$ and $w_1, w_2, \dots, w_{m-r}$ complete it to bases in U and W respectively, and the remaining vectors are complementary to $U + W$. The Plücker embedding (**??**) sends U and V to the different basis monomials

$$v_1 \wedge \cdots \wedge v_r \wedge u_1 \wedge \cdots \wedge u_{m-r} \neq v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_{m-r}$$

in $\Lambda^m V$.

Exrc. 5.2. (Comp. with general theory from n° 2.6 on p. 23.) The cone $C = P \cap T_p P$ consist of all lines passing through p and laying on P. On the other hand, it consists of all lines joining its vertex p with a smooth quadric $G = C \cap H$ cut out of C by any 3-dimensional hyperplane $H \subset T_p P$ complementary to p inside $T_p P \simeq \mathbb{P}_4$. Thus, any line on P passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and π_α, π_β are two planes spanned by p and two lines laying on the Segre quadric G and passing through p' (see fig. 5◇1 on p. 61).

Exrc. 5.4. See n° 5.3.3 on p. 67.

Exrc. 5.5. If $\omega \in P$, then $Z = T_\omega P$ and $\omega = \mathfrak{u}(\ell)$ for some lagrangian line $\ell \subset \mathbb{P}(V)$. Then all lines in $\mathbb{P}_3$ intersecting ℓ have to be lagrangian as well. This forces Ω to be degenerated.

Exrc. 5.6. The relations $w = e \cdot A_w^t$, $u = e \cdot A_u^t$, $w = u \cdot C_{uw}$, where e, u, w are the row matrices whose elements are the corresponding basis vectors, force $A_w^t = A_u^t C_{uw}$.

Exrc. 5.7. See Example 4.3 on p. 46.

Exrc. 5.8. Use the Plücker relation (4-47) on 57 and appropriate congruence reasons avoiding the complete enumeration of 720 matchings between a_{ij} and the given 6 numbers.

Exrc. 5.15. Since an alternating polynomial, considered as a polynomial in x_j with coefficients in the polynomial ring on the remaining variables, has the root $x_j = x_i$, it is divisible by $(x_i - x_j)$ for all $i \neq j$.

Exrc. 6.1. Let polynomials $f(x), g(x) \in I$ have degrees $m \geqslant n$ and leading coefficients a, b. Then $a + b$ equals either zero or the leading coefficient of polynomial $f(x) + x^{m-n} \cdot g(x) \in I$ of degree m. Similarly, for every $\alpha \in K$ the product αa either is zero or equals the leading coefficient of polynomial $\alpha f(x) \in I$ of degree m.

Exrc. 6.2. Repeat the arguments proving Theorem 6.1 on p. 70 but cancel non-zero monomials of the lowest degree instead of the leading.

Exrc. 6.3. Let $\pi : A \twoheadrightarrow B$ be the quotient epimorphism. The complete preimage $\pi^{-1}(I)$ of every ideal $I \subset B$ is an ideal in A, and therefore, it is generated by a finite set of element. Their images under π generate I.

Exrc. 6.4. Begin with $f_0 = z \sin(2\pi i z)$.

Exrc. 6.5. It is enough to construct such extension for just one monic irreducible polynomial $f \in B[x]$ of positive degree. If $\deg f = 1$, put $C = B$. Then use induction on $\deg f$. The quotient ring $D = B[x]/(f)$ contains B as the subring formed by residue classes of the constants. Write $\vartheta \in D$ for the residue class of x. Then $f(\vartheta) = 0$ and therefore, f is divisible by $(x - \vartheta)$ in $D[x]$, that is, becomes a product of irreducible monic polynomials of smaller degree in $D[x]$.

Exrc. 6.6. An element $a \in K \smallsetminus \mathfrak{m}$ is invertible in $K/\mathfrak{m}$ if and only if $1 \in (a, \mathfrak{m})$.