Projective spaces

AG1◇1. Let V be a vector space of dimension n over the finite field $\mathbb{F}_q$ of q elements. How many

a) bases **b)** subspaces of dimension k

are there in V? How many

c) affine **d)** projective

subspaces of dimension k are there in $\mathbb{A}(V)$ and $\mathbb{P}(V)$ respectively?

AG1◇2*. Write $G_n^k(q)$ for the rational function of q that gives the answer to prb. AG1◇1 (b). Compute $\lim_{q\to 1} G_n^k(q)$.

AG1◇3. Consider the projective closures of the following affine plane curves:

a) $y = x^2$ **b)** $y = x^3$ **c)** $y^2 + (x-1)^2 = 1$ **d)** $y^2 = x^2(x+1)$

Write down their homogeneous equations and their affine equations in two other standard affine charts on $\mathbb{P}_2$. Draw all these 12 affine curves.

AG1◇4 (Pythagorian triples). Write $(t_0 : t_1 : t_2)$ for homogeneous coordinates on $\mathbb{P}_2$. Let line ℓ and conic Q be given by the equations $t_2 = 0$ and $t_0^2 + t_1^2 = t_2^2$ respectively. Consider the point $O = (1 : 0 : 1) \in Q$. For every point $P = (p : q : 0) \in \ell$, find the coordinates of intersection point $Q \cap (OP)$ different from O. Show that the projection from O maps Q bijectively onto ℓ. Find a triple of quadratic polynomials $a(p,q)$, $b(p,q)$, $c(p,q)$ whose values produce all (up to common integer factor) integer solutions of the Pythagor equation $a^2 + b^2 = c^2$ as (p,q) runs through $\mathbb{Z} \times \mathbb{Z}$.

AG1◇5. Let the real Euclidian plane $\mathbb{R}^2$ be included in the complex projective plane $\mathbb{CP}_2$ as the real part[1] of the standard affine chart U_0.

a) Find two points on $\mathbb{CP}_2$ laying on every conic visible inside $\mathbb{R}^2$ as a circle.

b) Let a conic $C \subset \mathbb{CP}_2$ pass through these two points and have at least 3 non-collinear points visible inside $\mathbb{R}^2$. Prove that $C \cap \mathbb{R}^2$ is a circle.

AG1◇6 (Veronese map). Write $S^d V^*$ for the space of homogeneous polynomials of degree d on a vector space V. The Veronese map $v_d : V^* \to S^d V^*$ takes a linear form $\psi \in V^*$ to its d-th power $\psi^d \in S^d V^*$.

a) Find $\dim S^d V^*$, if $\dim V = n$.

b) Over a field of zero characteristic, prove that the image of v_d linearly spans the whole $S^d V^*$.

c) Is the same true for all d over a field of characteristic $p > 0$?

AG1◇7 (projections of twisted cubic). Write $\mathbb{P}_1 = \mathbb{P}(U^*)$ (resp. $\mathbb{P}_3 = \mathbb{P}(S^3 U^*)$) for the space of linear (resp. cubic) homogeneous forms in t_0, t_1 up to proportionality. The image of the Veronese map[2] $v_3 : \mathbb{P}_1 \hookrightarrow \mathbb{P}_3$ is called *twisted cubic* and denoted $C_3 \subset \mathbb{P}_3$. Describe the projection of C_3

a) from the point t_0^3 to the plane spanned by $3\,t_0^2 t_1$, $3\,t_0 t_1^2$, and t_1^3

b) from the point $3\,t_0^2 t_1$ to the plane spanned by t_0^3, $3\,t_0 t_1^2$, and t_1^3

c) from the point $t_0^3 + t_1^3$ to the plane spanned by t_0^3, $3\,t_0^2 t_1$, and $3\,t_0 t_1^2$.

Namely, write an explicit parametric representation for the target curve in appropriate coordinates on the target plane, then find affine and homogeneous equations for this curve. Find the degree of target curve. Plot affine parts of the curve in three different charts covering the target plane. Do they have self-intersections or cusps?

AG1◇8. Let $\dim V = n + 1$ and $f : V \xrightarrow{\sim} V$ be a linear isomorphism such that all fixed points of the induced map $\overline{f} : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V)$ are isolated. How many such fixed points may be there?

[1] That is, the set of points with real coordinates in the standard basis.

[2] Comp. with prb. AG1◇6.

Individual report card of ____ . Task 1 (September 6, 2017)
(write your name and surname)

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