Projective quadrics

AG2◇1. Write the polarization $\tilde{q}(A,B)$ of quadratic form $q(A) = \det A$ on the space $\mathrm{Mat}_2(\Bbbk)$ of 2×2-matrices as $\tilde{q}(A,B) = \frac{1}{2}\operatorname{tr}(AB^\vee)$. Describe explicitly how to get $B^\vee$ from B.

AG2◇2 (Euclidean polarities). Consider a circle on the real Euclidean plane $\mathbb{R}^2$. By means of the ruler and compasses, construct

a) the polar line of a given point laying inside the circle

b) the pole of a given line non-intersecting the circle.

AG2◇3. Show that all conics passing through the points $a = (1 : 0 : 0)$, $b = (0 : 1 : 0)$, $c = (0 : 0 : 1)$, $d = (1 : 1 : 1)$ in $\mathbb{P}_2$ form a pencil. Write an explicit equation for the conics of this pencil[1].

AG2◇4. Over an algebraically closed field, let a pencil of conics on $\mathbb{P}_2$ contain a smooth conic. Can this pencil contain exactly **a)** 0 **b)** 1 **c)** 2 **d)** 3 **e)** 4 different degenerated conics? Does there exist a pencil of conics on $\mathbb{P}_2$ without any smooth conics at all?

AG2◇5. Over an algebraically closed field, are there two smooth conics in $\mathbb{P}_2$ intersecting in exactly **a)** 1 **b)** 2 **c)** 3 different points?

AG2◇6. Show that the polar lines of a given point $a \in \mathbb{P}_2$ w.r.t. all the smooth conics in a given pencil are intersecting in one common point.

AG2◇7. Over the field $\mathbb{F}_9$ of nine elements[2], find the cardinality of

a) the conic $x_0^2 + x_1^2 + x_2^2 = 0$ in $\mathbb{P}_2$

b) the quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ in $\mathbb{P}_3$.

AG2◇8. Given 4 mutually non-intersecting lines in **a)** $\mathbb{P}(\mathbb{C}^4)$ **b)** $\mathbb{P}(\mathbb{R}^4)$ **c*)** $\mathbb{C}^3$ **d*)** $\mathbb{R}^3$, find how many lines do intersect them all. List all possible answers and indicate those which are stable under small perturbations of the given lines.

AG2◇9. Consider the space $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$ of conics in $\mathbb{P}_2 = \mathbb{P}(V)$. Write $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$ for the locus of singular conics. Show that

a) S is a cubic algebraic hypersurface

b) the set $\mathrm{Sing}(S)$ of singular points on S coincides with the image of quadratic Veronese embedding

$$v_2 : \mathbb{P}(V^*) \hookrightarrow \mathbb{P}_5, \quad \varphi \mapsto \varphi^2,$$

that is, a point $q \in S$ is singular iff the corresponding conic $Q = V(q) \subset \mathbb{P}_2$ is a double line

c) for a smooth point $q \in S$, which corresponds to a split conic $V(q) = \ell_1 \cup \ell_2 \subset \mathbb{P}_2$, the tangent space T_qS in $\mathbb{P}_5$ consists of all conics passing through the singular point $\ell_1 \cap \ell_2$ of $V(q)$ in $\mathbb{P}_2$.

[1]This should be a quadratic form whose coefficients depend linearly on two homogeneous parameters.

[2]Recall that $\mathbb{F}_9 = \mathbb{Z}[x]/(3, x^2+1)$ consists of elements $a + b\sqrt{-1}$, where $a, b \in \mathbb{F}_3 = \mathbb{Z}/(3)$ and $\sqrt{-1}\cdot\sqrt{-1} = -1 \in \mathbb{F}_3$.

Individual report card of ______________________________ (write your name and surname). Task 2 (September 14, 2017)

№	date	verified by	signature
1			
2			
3			
4			
5a			
b			
c			
6			
7a			
b			
8a			
b			
c			
d			
9a			
b			
c			