Tensors and Plücker – Segre – Veronese interaction

AG3⋄1. Are the following decompositions valid for any vector space V over a field of zero characteristic:
a) $V^{\otimes 2} \simeq \operatorname{Sym}^2 V \oplus \operatorname{Alt}^2 V$ **b)** $V^{\otimes 3} \simeq \operatorname{Sym}^3 V \oplus \operatorname{Alt}^3 V$?
If yes, prove it. If no, give an explicit example of a tensor that can not be decomposed in this way.

AG3⋄2 (spinor decomposition). Let $V = \operatorname{End}(U)$, where $\dim U = 2$, $\operatorname{char} \Bbbk \neq 2$. Show that

$$V^{\otimes 2} \simeq \underbrace{\left((S^2U^* \otimes S^2U) \oplus \left(\Lambda^2U^* \otimes \Lambda^2U\right) \right)}_{\simeq \operatorname{Sym}^2 V} \bigoplus \underbrace{\left((S^2U^* \otimes \Lambda^2U) \oplus \left(\Lambda^2U^* \otimes S^2U\right) \right)}_{\simeq \operatorname{Alt}^2 V} .$$

HINT. Use the decompositions $V = U^* \otimes U$ and $U^{\otimes 2} \simeq S^2U \oplus \Lambda^2U$.

AG3⋄3. For finite dimensional vector spaces U, V construct the canonical linear isomorphisms

$$\operatorname{Hom}\big(U \otimes \operatorname{Hom}(U,W)\,,\, W\big) \simeq \operatorname{End}\big(\operatorname{Hom}(U,W)\big) \simeq \operatorname{Hom}\big(U\,,\, W \otimes \operatorname{Hom}(U,W)^*\big).$$

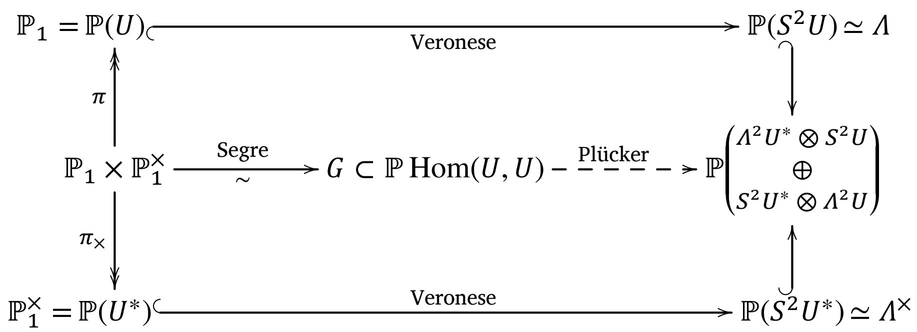
Write $c : U \otimes \operatorname{Hom}(U,W) \to W$ for the linear map $u \otimes f \mapsto f(u)$ and $\tilde{c} : U \to \operatorname{Hom}(U,W)^* \otimes W$ for the linear map corresponding to c under the above isomorphism. May $\tilde{c}$ be non-injective? Describe the linear endomorphism of $\operatorname{Hom}(U,W)$ corresponding to c and $\tilde{c}$.
HINT. Use the isomorphism $\operatorname{Hom}(A,B) \simeq A^* \otimes B$, and that the decomposable tensors linearly span $A^* \otimes B$.

AG3⋄4. Let $G = V(g) \subset \mathbb{P}_3 = \mathbb{P}(V)$ be a smooth quadric. Write $\widetilde{g}$ for the polarization of quadratic form g and $\Lambda^2\widetilde{g}$ for the bilinear form on $\Lambda^2 V$ defined by prescription

$$\Lambda^2\widetilde{g}(\, v_1 \wedge v_2 \,,\, w_1 \wedge w_2 \,) \overset{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1,w_1) & \widetilde{g}(v_1,w_2) \\ \widetilde{g}(v_2,w_1) & \widetilde{g}(v_2,w_2) \end{pmatrix} .$$

a) Verify that $\Lambda^2\widetilde{g}$ is symmetric and non degenerate. Write its Gram matrix in the basis $e_i \wedge e_j$ build from an orthonormal basis e_1, e_2, e_3, e_4 for g in V.
b) Prove that the quadric $V(\Lambda^2 g) \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ intersects the Plücker quadric $\operatorname{Gr}(2,V) \subset \mathbb{P}_5$ along the set of all tangent lines to $G \subset \mathbb{P}_3$.

AG3⋄5. Consider the previous prb. AG3⋄4 for the space $V = \operatorname{End}(U)$ from prb. AG3⋄2 and $g = \det$, that is, for the Segre quadric $G = V(\det) \subset \mathbb{P}(V)$. Show that two families of ruling lines on G are mapped by the Plücker embedding $\operatorname{Gr}(2,V) \hookrightarrow \mathbb{P}(\Lambda^2 V)$ to the pair of smooth plane conics cut out the Plücker quadric by the complementary planes $\Lambda = \mathbb{P}(\Lambda^2U^* \otimes S^2U)$, $\Lambda^\times = \mathbb{P}(S^2U^* \otimes \Lambda^2U)$ embedded into $\mathbb{P}\big(\Lambda^2 \operatorname{End}(U)\big)$ via prb. AG3⋄2. Verify that the both conics are embedded into these planes by the Veronese maps, i.e., the following diagram[1] is commutative:



AG3⋄6. Write $S \subset \mathbb{P}_3$ for the surface ruled by the tangent lines to the Veronese cubic. Write an explicit equation for S, find its degree and all the singular points on S.

[1] The Plücker embedding is dashed, because it sends lines to points.

Individual report card of ____. Task 3 (October 5, 2017)
(write your name and surname)

№	date	verified by	signature
1a			
b			
2			
3			
4a			
b			
5			
6			