Examples of curves

AG4◇1 (rational normal curves). Consider the vector space U with basis t_0, t_1 and use the coefficients a_i of expansion $f(t_0,t_1) = \sum_{n=0}^{d} a_n \cdot \binom{d}{n} t_0^n t_1^{d-n}$ as homogeneous coordinates in $\mathbb{P}_d = \mathbb{P}(S^d U)$. Show that the images of maps $\nu, \varphi, \psi : \mathbb{P}_1 \hookrightarrow \mathbb{P}_d$ listed below are transformed one to another by appropriate linear automorphisms of $\mathbb{P}_d$: **a)** $\nu : f \mapsto f^d$ for all $f \in \mathbb{P}_1$ (the *Veronese map* of degree d)

b) $\varphi : a \mapsto (f_0(a) : f_1(a) : \dots : f_d(a))$, where $f_0, f_1, \dots, f_m$ are linearly independent homogeneous polynomials of degree d in $a = (a_0, a_1)$

c) $\psi : a \mapsto \left(\det^{-1}(p_0, a) : \det^{-1}(p_1, a) : \cdots : \det^{-1}(p_d, a)\right)$, where $p_0, p_1, \dots, p_d \in \mathbb{P}_1$ are some fixed mutually different points, and $\det(a,b) \overset{\text{def}}{=} a_0 b_1 - a_1 b_0$ for $a = (a_0 : a_1)$, $b = (b_0 : b_1)$.

AG4◇2. In the notations of prb. AG3◇1, show that every linear automorphism of $\mathbb{P}_d$ sending the Veronese curve $\nu(\mathbb{P}_1)$ to itself is induced by a linear change of variables t_0, t_1, i.e., by a linear automorphism of $\mathbb{P}_1$.

AG4◇3. Given $d+3$ points $p_1, p_2, \dots, p_d$, a, b, $c \in \mathbb{P}_d$ such that any $(d+1)$ of them do not lie in a hyperplane, write $\ell_i \simeq \mathbb{P}_1$ for the pencil of hyperplanes passing through all p_νs but p_i, and $\psi_{ij} : \ell_j \simeq \ell_i$ for the homography sending the triple of hyperplanes in ℓ_j passing through a, b, c to the similar triple in ℓ_i. Show that $\bigcup_{H \in \ell_1} H \cap \psi_{21}(H) \cap \dots \cap \psi_{n1}(H)$ is a rational normal curve from prb. AG3◇1, and this is the unique rational normal curve passing through the given $d+3$ points.

AG4◇4 (rational curves). A curve $C = V(f) \subset \mathbb{P}_2$ is called *rational* if there are three homogeneous polynomials $p_0, p_1, p_2 \in \Bbbk[t_0, t_1]$ of the same degree such that the map $\mathbb{P}_1 \to \mathbb{P}_2$, $\alpha \mapsto (p_0(\alpha) : p_1(\alpha) : p_2(\alpha))$, establishes a bijection between $\mathbb{P}_1$ and C. Show that $\deg f = \deg p_i$ in this case, and prove that every rational curve of degree d in $\mathbb{P}_2$ is a plane projection of the Veronese curve $C_d \subset \mathbb{P}_d$.

AG4◇5. Describe intersection multiplicities at the origin in $\mathbb{A}^2$ between the curve $x^2 y + x y^2 = x^4 + y^4$ and every line passing trough the origin. Find all singular points on the projective closure of this curve over an algebraically closed field.

AG4◇6. Find all singular points and compute the intersection multiplicities with all lines passing through these points[1] for the projective plane curve given by **a)** homogeneous equation $(x_0 + x_1 + x_2)^3 = 27\, x_0 x_1 x_2$

b) affine equation $(x^2 - y + 1)^2 = y^2(x^2 + 1)$.

AG4◇7 (plane cubics). A *plane cubic* is a curve of degree 3 in $\mathbb{P}_2$ over algebraically closed field.

a) How many singular points may a plane cubic have, and what can be their multiplicities?

b) Up to a linear projective automorphism of $\mathbb{P}_2$, classify all reducible plane cubics split into a union of lines or a line and a conic.

c) Up to a linear projective automorphism of $\mathbb{P}_2$, classify all rational plane cubics.

d) Show that every singular plane cubic is rational, but every smooth is not.

HINT. Use the projection from the singular point onto a line.

e) How many tangent lines to a smooth plane cubic can be drawn from a generic point of $\mathbb{P}_2$?

f) How many inflection points[2] are there on a smooth plane cubic?

g) For a smooth plane cubic C and an inflection point $a \in C$, show that there are exactly 3 non-inflection tangent lines to C drawn from a, and they touch C in a triple of collinear points.

HINT. Show that the quadratic polar of a is a split conic smooth at a.

h*) Deduce from the previous result that every smooth plane cubic is described in appropriate affine coordinates by the equation $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \Bbbk$.

i*) Show that two smooth plane cubics can be transformed one to another by a linear projective automorphism of $\mathbb{P}_2$ iff their *j-invariants* $j(\lambda) = 2^8(\lambda^2 - \lambda + 1)^3 / (\lambda(\lambda - 1))^2$, where λ is as above, coincide.

HINT. Show that $\Bbbk(j) \subset \Bbbk(\lambda)$ is the field of invariants for the action of the symmetric group S_3 on $\Bbbk(\lambda)$ by linear fractional transformations of λ permuting the values $\lambda = \infty$, 0, 1.

AG4◇8. Find all lines laying on the projective cubic surface given by **a)** the affine equation $xyz = 1$

b*) (Fermat's cubic) the homogeneous equation $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$.

[1] Separately for every singular point.

[2] A point p of a curve $C \subset \mathbb{P}_2$ is called an *inflection point* if the tangent line $T_p C$ intersects C at p with the multiplicity at least 3.

Individual report card of ____________________________________. Task 4 (October 12, 2017)
(write your name and surname)

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