Commutative algebra and affine algebraic geometry

AG5◇1. Let a polynomial f vanish along a hypersurface $V(g) \subset \mathbb{A}^n$ over an algebraically closed field. Prove that every irreducible factor of g divides f.

AG5◇2. For the ideal $J = (xy, yz, zx) \subset \mathbb{C}[x, y, z]$, describe $V(J) \subset \mathbb{A}^3$ and $I(V(J)) \subset \mathbb{C}[x, y, z]$. Is it possible to describe the same variety by two polynomial equations?

AG5◇3. Find $f \in I(V(J)) \setminus J$ for the ideal $J = (x^2 + y^2 - 1, y - 1) \subset \mathbb{C}[x, y]$.

AG5◇4. Describe $V(J) \subset \mathbb{A}^3$ and $I(V(J)) \subset \mathbb{C}[x, y, z]$ for the ideals
a) $J = (xy, (x - y)z)$ **b)** $J = (xy + yz + zx, x^2 + y^2 + z^2)$.

AG5◇5. Show that the ideal $I(C_3) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ generated by all homogeneous polynomials vanishing identically on the Veronese cubic $C_3 \subset \mathbb{P}_3$
a) is generated by three quadratic polynomials **b*)** can not be generated by two polynomials.

AG5◇6. For an arbitrary field $\mathbb{k}$ and two ideals $I, J \subset \mathbb{k}[x_1, x_2, \dots, x_n]$, is it true or not that
a) $\sqrt{IJ} = \sqrt{I \cap J}$ **b)** $\sqrt{IJ} = \sqrt{I}\sqrt{J}$ **c)** $\left(I = \sqrt{I} \;\&\; J = \sqrt{J}\right) \Rightarrow IJ = \sqrt{IJ}$?

AG5◇7. Let $B \supset A$ be an extension of commutative rings such that B is finitely generated as A-module. Prove that $\mathfrak{m}B \neq B$ for all maximal ideals $\mathfrak{m} \subset A$.

AG5◇8 (Zariski topology).
a) Let A be a finitely generated reduced $\mathbb{k}$-algebra, $X = \operatorname{Spec}_{\mathrm{m}} A$ the set of its maximal ideals. Verify that the sets $V(I) = \{ x \in X \mid f(x) = 0 \quad \forall f \in I \}$, where $I \subset A$ is an ideal, satisfy the axioms for closed sets of a topology. That is, the sets X, $\varnothing$, $V(I) \cup V(J)$ for any two ideals $I, J \subset A$, and $\bigcap_{I \in M} V(I)$ for any set M of ideals in A, can be written as $V(K)$ for appropriate ideal $K \subset A$.
b) The same question for an arbitrary commutative ring A, the set $X = \operatorname{Spec} A$ of all prime ideals[1] $\mathfrak{p} \subset A$, and the subsets $V(I) = \{\mathfrak{p} \in X \mid I \subset \mathfrak{p}\}$, where $I \subset A$ is an ideal.

AG5◇9. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic varieties. Assuming that equations for X, Y are known, write explicit equations for $X \times Y \subset \mathbb{A}^{n+m}$ and show that $X \times Y$ is irreducible for irreducible X, Y.

AG5◇10. Let a $\mathbb{k}$-algebra A be of finite dimension as a vector space over $\mathbb{k}$. Prove that $\operatorname{Spec}_{\mathrm{m}}(A)$ is finite.

AG5◇11. Give an example of regular non-finite[2] morphism of affine algebraic varieties $f : X \to Y$ such that every non-empty fiber of f is finite.

AG5◇12. For a given affine hypersurface $S = V(f) \subset \mathbb{A}^n$ over an algebraically closed field, describe all vectors of the form $v = (1, v_1, v_2, \dots, v_n)$ such that the parallel projection of S along v to the hyperplane $x_1 = 0$ is
a) dominant[3] **b)** surjective **c)** finite.

AG5◇13. For an affine hypersurface $S \subset \mathbb{A}^n$ over an algebraically closed field, prove that
a) the central projection of S from a point $p \notin S$ onto a hyperplane $H \not\ni p$ is dominant
b) there exists a finite surjective parallel projection of S onto a hyperplane.

AG5◇14*. For a normal[4] commutative ring A, prove that $A[x]$ is normal as well.

[1] A proper ideal $\mathfrak{p} \subset A$ is called to be *prime* if the quotient ring $A/\mathfrak{p}$ has no zero divisors.
[2] A regular morphism of affine varieties $f : X \to Y$ is called to be *finite* if $\mathbb{k}[X]$ is finitely generated as a $f^*(\mathbb{k}[Y])$-module.
[3] A morphism $f : X \to Y$ is called *dominant* if $f(X_i)$ is dense in Y for every irreducible component $X_i \subset X$.
[4] That is, having no zero divisors and integrally closed within its field of fractions.

Individual report card of ______________________________ . Task 5 (October 19, 2017)
(write your name and surname)

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