Dimensions of Algebraic Manifolds

AG7◇1. Prove that $\dim_{(x,y)}(X \times Y) = \dim_x X + \dim_y Y$ at every point $(x, y) \in X \times Y$.

AG7◇2. Let $X \subset \mathbb{P}_n = \mathbb{P}(V)$ be a projective variety of dimension d. Show that projective subspaces $H \subset \mathbb{P}(V)$ of dimension $(n-d)$ intersecting X in a finite number of points form a Zariski open subset in the grassmannian[1] $\mathrm{Gr}(n+1-d, V)$.

HINT. Use the projection of the incidence graph $\Gamma = \{(x, H) \in X \times \mathrm{Gr}(n+1-d, V) \mid x \in H\}$ onto X to show that Γ is an irreducible projective variety and find $\dim \Gamma$. Then analyze the second projection $\Gamma \to \mathrm{Gr}(n+1-d, V)$.

AG7◇3 (resultant). Given positive integers $d_0, d_1, \dots, d_n$, let $\mathbb{P}_{N_i} = \mathbb{P}\left(S^{d_i} V^*\right)$ for $0 \leqslant i \leqslant n$ and $V = \Bbbk^{n+1}$. Show that:

a) $\Gamma \stackrel{\text{def}}{=} \{(S_0, S_1, \dots, S_n, p) \in \mathbb{P}_{N_0} \times \dots \times \mathbb{P}_{N_n} \times \mathbb{P}_n \mid p \in S_0 \cap S_1 \cap \dots \cap S_n\}$ is an irreducible projective variety, and find $\dim \Gamma$

b) up to a scalar factor, there exists a unique irreducible polynomial R in coefficients of homogeneous polynomials $f_0, f_1, \dots, f_n$ of degrees $d_0, d_1, \dots, d_n$ in $n+1$ variables such that a given system of $n+1$ equations $f_\nu = 0$ has a non-zero solution if and only if the polynomial R vanishes at the coefficients of these F_ν's.

AG7◇4 (geometric definition of dimension). Show that the dimension of an irreducible variety $X \subset \mathbb{P}_n$ equals: **a)** the maximal $d \in \mathbb{Z}$ such that $X \cap L \neq \varnothing$ for every dimension $(n-d)$ projective subspace $L \subset \mathbb{P}_n$ **b)** the minimal $d \in \mathbb{Z}$ for which there is an $(n-d-1)$-dimensional projective subspace $L \subset \mathbb{P}_n$ such that $X \cap L = \varnothing$ **c)** the minimal $d \in \mathbb{Z}$ such that $X \cap L = \varnothing$ for a generic[2] dimension $(n-d-1)$ projective subspace $L \subset \mathbb{P}_n$.

AG7◇5. Show that there exists a unique homogeneous polynomial P on the space of homogeneous forms of degree 4 in 4 variables such that P vanishes at f iff the surface $V(f) \subset \mathbb{P}_3$ contains a line.

HINT. Show that the incidence graph $\Gamma = \{(\ell, S) \in \mathrm{Gr}(2,4) \times \mathbb{P}(S^4(\mathbb{C}^4)^*) \mid \ell \subset S\}$ is a projective variety and use the projection $\Gamma \to \mathrm{Gr}(2,4)$ to show that Γ is irreducible and find $\dim \Gamma$. Then find a finite nonempty fiber for the second projection $\Gamma \to \mathbb{P}(S^4(\mathbb{C}^4)^*)$.

AG7◇6. Show that the image of a regular dominant morphism contains an open dense subset.

AG7◇7. Show that lines lying on a smooth odd dimensional quadric $Q \subset \mathbb{P}_{2n}$ form an irreducible projective variety and find its dimension.

AG7◇8. Let $\varphi : X \to Y$ be a regular morphism of algebraic manifolds. Show that isolated[3] points of fibers $\varphi^{-1}(y)$ draw an open subset of X when y runs through Y.

HINT. Use Chevalley's theorem on semi-continuity from the Lecture Notes.

AG7◇9* (Chevalley's constructivity theorem). Prove that an image of any regular morphism of algebraic varieties is *constructive*, i.e., can be constructed from a finite number of open and closed subsets by a finite number of unions, intersections, and taking complements.

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[1] This grassmannian parameterizes all subspaces of dimension $(n-d)$ in $\mathbb{P}(V)$.

[2] That is, taken from some Zariski open dense subset of grassmannian $\mathrm{Gr}(n-d, V)$, which parametrizes all dimension $(n-d-1)$ projective subspaces in $\mathbb{P}(V)$.

[3] A point $p \in M$ is called *isolated* point of a subset $M \subset X$ in a topological space X, if it has an open neighborhood $U \ni p$ such that $U \cap M = \{p\}$.

Individual report card of ______________________________________. Task 7 (November 23, 2017)
(write your name and surname)

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