§1 General Nonsense

1.1 Categories. A category $\mathcal{C}$ consists of *a class*[1] of objects $\operatorname{Ob}\mathcal{C}$, where any ordered pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$ is equipped with *a set* of morphisms from X to Y

$$\operatorname{Hom}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)\,.$$

It is convenient to think of the morphisms from X to Y as arrows $\varphi : X \to Y$. The sets $\operatorname{Hom}(X,Y)$ are disjoint for distinct pairs X, Y and their union over all $X, Y \in \operatorname{Ob}\mathcal{C}$ is denoted $\operatorname{Mor}\mathcal{C} = \bigsqcup_{X,Y}\operatorname{Hom}_{\mathcal{C}}(X,Y)$. For each ordered triple $X, Y, Z \in \operatorname{Ob}\mathcal{C}$ there is a composition map[2]

$$\operatorname{Hom}(Y,Z)\times\operatorname{Hom}(X,Y)\to\operatorname{Hom}(X,Z)\,,\quad (\varphi,\psi)\mapsto\varphi\circ\psi\quad (=\varphi\psi)\,, \tag{1-1}$$

which is associative: $(\chi\circ\varphi)\circ\psi = \chi\circ(\varphi\circ\psi)$ each time when LHS or RHS is defined. Finally, each object $X \in \operatorname{Ob}\mathcal{C}$ has *the identity endomorphism*[3] $\operatorname{Id}_X \in \operatorname{Hom}(X,X)$ such that $\varphi\circ\operatorname{Id}_X = \varphi$ and $\operatorname{Id}_X\circ\psi = \psi$ for all arrows $\varphi : X \to Y$ and $\psi : Z \to X$.

A subcategory $\mathcal{D} \subset \mathcal{C}$ is a category whose objects, arrows, and compositions come from $\mathcal{C}$. A subcategory $\mathcal{D} \subset \mathcal{C}$ is called *full*, if $\operatorname{Hom}_{\mathcal{D}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \operatorname{Ob}\mathcal{D}$.

A category is called *small*, if $\operatorname{Ob}\mathcal{C}$ is *a set*. In this case $\operatorname{Mor}\mathcal{C}$ is a set as well.

Example 1.1 (non-small categories)
The following categories often appear in examples and are not small: category $\mathcal{S}et$ of all sets and all mapping between them, category $\mathcal{T}op$ of all topological spaces and continuous mappings, category $\mathcal{V}ec_{\Bbbk}$ of vector spaces over a field $\Bbbk$ and $\Bbbk$-linear mappings, its full subcategory $vec_{\Bbbk}$ formed by finite dimensional spaces, categories R-$\mathcal{M}od$ and $\mathcal{M}od$-R of left and right modules over a ring R and R-liner mappings, their full subcategories R-mod and mod-R formed by finitely presented[4] modules, category $\mathcal{A}b = \mathbb{Z}$-$\mathcal{M}od$ of abelian groups and category $\mathcal{G}rp$ of all groups and group homomorphisms, category $\mathcal{C}mr$ of commutative rings with unities and ring homomorphisms sending unity to unity, etc.

Example 1.2 (posets)
Each poset[5] M is a category whose objects are the elements $m \in M$ and

$$\operatorname{Hom}_M(n,m) = \begin{cases}\text{one element, if } n \leqslant m\\ \varnothing \quad \text{otherwise.}\end{cases}$$

The composition of arrows $k \leqslant \ell$ and $\ell \leqslant n$ is the arrow $k \leqslant n$. Most important for us special example of such a category is a category $\mathcal{U}(X)$ of all open subsets in a topological space X and inclusions as the morphisms:

$$\operatorname{Hom}_{\mathcal{U}(X)}(U,W) = \begin{cases}\text{the inclusion } U \hookrightarrow W\text{, if } U \subseteq W\\ \varnothing\,, \quad \text{if } U \not\subseteq W.\end{cases}$$

[1]We would not like to formalize here this logical notion explicitly (see any ground course of Math Logic). However we will consider e.g. *the category of sets* whose objects — sets — do not form a set.

[2]like the multiplication symbol, the composition symbol « ∘ » is usually skipped

[3]it is unique because of $\operatorname{Id}' = \operatorname{Id}'\circ\operatorname{Id}'' = \operatorname{Id}''$

[4]a module is called *finitely presented*, if it is isomorphic to a quotient of a finitely generated free module through its finitely generated submodule

[5]that is, partially ordered set

Example 1.3 (small categories vs associative algebras)
Each associative algebra A with unity $e \in A$ over a commutative ring K is a category with just one object e and $\mathrm{Hom}(e, e) = A$, where the composition of arrows equals the product in A. Vice versa, associated with an arbitrary small category $\mathcal{C}$ and a commutative ring K is an associative algebra $K[\mathcal{C}]$ formed by all formal finite linear combinations of morphisms in $\mathcal{C}$ with coefficients in K:

$$K[\mathcal{C}] = \bigoplus_{X,Y \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Hom}(X, Y) \otimes K = \left\{ \sum x_i \varphi_i \,\middle|\, \varphi_i \in \mathrm{Mor}(\mathcal{C}),\ x_i \in K \right\},$$

where we write $M \otimes K$ for the free K-module with basis[1] M. The multiplication of arrows in $K[\mathcal{C}]$ is defined by the rule

$$\varphi\psi \overset{\text{def}}{=} \begin{cases} \varphi \circ \psi & \text{if the target of } \psi \text{ coincides with the source of } \varphi \\ 0 & \text{otherwise} \end{cases}$$

and is extended linearly onto arbitrary finite linear combinations of arrows. One can think of $K[\mathcal{C}]$ as an algebra of (maybe infinte) square matrices whose cells are numbered by the pairs of objects of category $\mathcal{C}$, an element from (Y, X)-cell belongs to free module $\mathrm{Hom}(X, Y) \otimes K$, and only finitely many such elements are non-zero. In general, algebra $K[\mathcal{C}]$ is non-commutative and without unity. However for each $f \in K[\mathcal{C}]$ there is an idempotent $e_f = e_f^2$ such that

$$e_f \circ f = f \circ e_f = f$$

(e.g. $\sum_X \mathrm{Id}_X$, where X runs through the sources and targets of all arrows that appear in f).

Example 1.4 (combinatorial simplexes)
Let Δ_{big} be the category of all finite ordered sets and order preserving maps[2]. This category is not small. However it contains a small full subcategory $\Delta \subset \Delta_{\text{big}}$ formed by the sets of integers

$$[n] \overset{\text{def}}{=} \{0, 1, \dots, n\}, \quad n \geqslant 0, \tag{1-2}$$

with their standard orderings. The ordered set (1-2) is called *the combinatorial n-simplex*. Category Δ is called *the simplicial category*.

Exercise 1.1. Show that algebra $\mathbb{Z}[\Delta]$ is generated by the arrows

$$e_n = \mathrm{Id}_{[n]} \qquad \text{(the identity endomorphism)} \tag{1-3}$$

$$\partial_n^{(i)} : [n-1] \hookrightarrow [n] \qquad \text{(the inclusion whose image does not contain } i\text{)} \tag{1-4}$$

$$s_n^{(i)} : [n] \twoheadrightarrow [n-1] \qquad \text{(the surjection sending } i \text{ and } (i+1) \text{ to the same element)} \tag{1-5}$$

and describe the generating relations[3] between these arrows.

[1]this module is formed by all finite formal linear combinations of elements of the set M with coefficients in K

[2]i.e. $\varphi : X \to Y$ such that $x_1 \leqslant x_2 \Rightarrow \varphi(x_1) \leqslant \varphi(x_2)$

[3]i.e. generators of the kernel of the canonical surjection from the free associative algebra generated by symbols e_n, $\partial_n^{(i)}$, $\partial_n^{(i)}$ onto algebra $\mathbb{Z}[\Delta]$

1.1.1 Mono, epi, and isomorphisms. A morphism φ in a category $\mathcal{C}$ is called *a monomorphism*[1] (resp. *an epimorphism*[2]), if it admits left (resp. right) cancellation, that is

$$\varphi\alpha = \varphi\beta \Rightarrow \alpha = \beta \quad (\text{resp. } \alpha\varphi = \beta\varphi \Rightarrow \alpha = \beta\,).$$

A morphism $\varphi : X \to Y$ is called *an isomorphism*[3], if there is a morphism $\psi : Y \to X$ such that $\varphi\psi = \mathrm{Id}_Y$ and $\psi\varphi = \mathrm{Id}_X$. In this case objects X and Y are called *isomorphic*. We depict injective, surjective, and invertible arrows as $\hookrightarrow$, $\twoheadrightarrow$, and $\xrightarrow{\sim}$ respectively.

Exercise 1.2. Find the cardinality of $\mathrm{Hom}_\Delta([n],[m])$. How many injective, surjective, and isomorphic arrows are there in $\mathrm{Hom}_\Delta([n],[m])$?

1.1.2 Rewersal of arrows. Associated with a category $\mathcal{C}$ is *an opposite category* $\mathcal{C}^{\mathrm{opp}}$ with the same objects but rewersed arrows, that is

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X,Y) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(Y,X) \quad \text{and} \quad \varphi^{\mathrm{opp}} \circ \psi^{\mathrm{opp}} = (\psi\circ\varphi)^{\mathrm{opp}}\,.$$

In terms of algebras, algebra $K[\mathcal{C}^{\mathrm{opp}}] = K[\mathcal{C}]^{\mathrm{opp}}$ is an opposite algebra of $K[\mathcal{C}]$. Injections in $\mathcal{C}$ become surjections in $\mathcal{C}^{\mathrm{opp}}$ and vice versa.

1.2 Functors. A *functor*[4] $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping

$$\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}\,, \quad X \mapsto F(X)\,,$$

and a collection of maps[5]

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))\,, \quad \varphi \mapsto F(\varphi)\,, \tag{1-6}$$

such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all $X \in \mathrm{Ob}\,\mathcal{C}$ and $F(\varphi\circ\psi) = F(\varphi)\circ F(\psi)$ each time when composition $\varphi\circ\psi$ is defined. In terms of algebras, a functor is a homomorfism of algebras $F : K[\mathcal{C}] \to K[\mathcal{D}]$. If all the maps (1-6) are surjective, functor F is called *full*. An image of a full functor is a full subcategory. If all the maps (1-6) are injective, F is called *faithful*. A faithful functor produces an injective homomorphism of algebras $F : K[\mathcal{C}] \to K[\mathcal{D}]$.

The simplest examples of functors are provided by *the identity functor* $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ acting identically on the objects and on the arrows and by *the forgetting functors*, sending categories of sets with extra structures and the morphisms respecting these structures[6] to the category $\mathcal{S}et$, of sets, by forgetting the structure.

Example 1.5 (geometric realization of combinatorial simplexes)
The geometric realization functor $\Delta \to \mathcal{T}op$ takes n-dimensional combinatorial simplex $[n]$ from (1-2) to the standard regular n-simplex[7]

$$\Delta^n = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum x_\nu = 1\,,\ x_\nu \geqslant 0 \right\} \subset \mathbb{R}^{n+1}\,, \tag{1-7}$$

[1] or *an injection*
[2] or *a surjection*
[3] or *an invertible morphism*
[4] or a *covariant* functor
[5] one map for each ordered pair $X, Y \in \mathrm{Ob}\,\mathcal{C}$
[6] e.g. topological spaces with continuous maps or vector spaces with linear maps
[7] that is the convex hull of the ends of the standard basic vectors $e_0, e_1, \dots, e_n \in \mathbb{R}^{n+1}$

and takes each order preserving map $\varphi : [n] \to [m]$ to the affine linear map $\varphi_* : \Delta^n \to \Delta^m$ that acts on the basic vectors as $e_\nu \mapsto e_{\varphi(\nu)}$. This is faithful but non-full functor. It sends generators (1-4), (1-5) of algebra $\mathbb{Z}[\Delta]$ to the i-th face inclusion $\Delta^{(n-1)} \hookrightarrow \Delta^n$ and to the i-th edge contraction[1] $\Delta^n \twoheadrightarrow \Delta^{(n-1)}$.

1.2.1 Presheaves. A functor $F : \mathcal{C}^{\text{opp}} \to \mathcal{D}$ is called *a contra-variant functor* from $\mathcal{C}$ to $\mathcal{D}$ or *a presheaf* of objects of category $\mathcal{D}$ on a category $\mathcal{C}$. It reverses the compositions $F(\varphi \circ \psi) = F(\psi) \circ F(\varphi)$. In terms of algebras, a contravariat functor produces an *anti*-homomorphism of algebras $K[\mathcal{C}] \to K[\mathcal{D}]$.

Example 1.6 (presheaves and sheaves of sections)
The notion «presheaf» has appeared initially in a context of the category $\mathcal{C} = \mathcal{U}(X)$ of open subsets $U \subset X$ in a given topological space X. A presheaf $F : \mathcal{U}(X)^{\text{opp}} \to \mathcal{D}$ attaches an object $F(U) \in \operatorname{Ob}\mathcal{D}$ to each open set $U \subset X$. This object is called (an object of) *sections* of F over U. Depending on $\mathcal{D}$, the sections can form a ring, an algebra, a vector space, a topological space, etc. Attached to an inclusion of open sets $U \subset W$ is a map $F(W) \to F(U)$ called *the restriction of sections* from W onto $U \subset W$. The restriction of a section $s \in F(W)$ onto a subset $U \subset W$ is usually denoted by $s|_U$. Here are some typical examples of such presheaves:

1) Presheaf Γ_E of the sets of local sections of a continuous mapping $p : E \to X$ has $\Gamma_E(U)$ equal to a set of maps $s : U \to E$ such that[2] $p \circ s = \operatorname{Id}_U$. Its restriction maps take sections to their restrictions onto smaller subsets.

2) Specializing the previous example to projection $p : X \times Y \to X$, we get the sheaf $\mathcal{C}^0(X, Y)$ of locally defined continuous mappings $s : U \to Y$.

3) Further specialization of the above examples leads to so called *structure presheaves* $\mathcal{O}_X$ such as the presheaf of local smooth functions $U \to \mathbb{R}$ on a smooth manifold X, or the presheaf of local holomorphic functions $U \to \mathbb{C}$ on a complex analytic manifold X, or the presheaf of local rational functions $U \to \Bbbk$ on an algebraic manifold X over a field $\Bbbk$ etc. All these presheaves are presheaves of algebras over the corresponding field $\mathbb{R}$, $\mathbb{C}$, or $\Bbbk$.

4) A constant presheaf S has $S(U)$ equal to a fixed set S for all open $U \subset X$ and all its restriction maps are the identity morphisms Id_S.

A presheaf F of sets on S is called *a sheaf*, if for any open W, any covering of W by open $U_i \subset W$, and any collection of sections $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j there exist a unique section $s \in F(W)$ such that $s|_{U_i} = s_i$ for all i. If there exist at most one such a section s but it does not have to exist, then F is called a *separable* presheaf. All above presheaves (1) – (4) are separable and only the last of them is not a sheaf, because for disjoint union $W = U_1 \sqcup U_2$ of open U_1, U_2 not any pair of constants $s_i \in S(U_i)$ appears as the restriction of some constant $s \in S(W)$. However, besides the constant presheaf S, associated to an arbitrary set S is

5) a *constant sheaf* $S^\sim$ whose sets of sections $S^\sim(U)$ consist of *continuous* maps $U \to S$, where S is considered with the *discrete* topology.

[1] i.e. projection onto a face along the edge joining i-th and $(i+1)$-th vertexes
[2] i.e. sending each point $x \in U$ to the fiber $p^{-1}(x) \subset E$ over x

Exercise 1.3. Find all antiderivatives[1] of real function $x \mapsto 1/x$.

Exercise 1.4. Show that the category of sheaves $\mathcal{Sh}(X)$ is a full subcategory of the category of presheaves $p\mathcal{Sh}(X)$.

Example 1.7 (triangulated topological spaces)
Write $\Delta_s \subset \Delta$ for non-full subcategory with $\mathrm{Ob}\,\Delta_s = \mathrm{Ob}\,\Delta$ and *injective*[2] order preserving maps as the morphisms. Category Δ_s is called the *semisimplicial* category.

Exercise 1.5. Show that algebra $K[\Delta_s]$ is generated by the identical arrows $e_n = \mathrm{Id}_{[n]}$ and the inclusions $\partial_n^{(i)}$ from (1-4).

A presheaf of sets $X : \Delta_s^{\mathrm{opp}} \to \mathcal{S}et$ on Δ_s is called a *semisimplicial set*. Each semisimplicial set is nothing but a combinatorial description for some *triangulated* topological space denoted by $|X|$ and called a *geometric realization* of semisimplicial set X. Namely, F attaches a set $X_n = X([n])$ to each non-negative integer n. Let us interpret the points $x \in X_n$ as disjoint regular n-simplexes Δ_x^n. The morphisms $\varphi : [n] \to [m]$ in category Δ_s stay in bijection with n-dimensional faces of regular m-simplex Δ^m. A map $X(\varphi) : X_m \to X_n$, which corresponds to such a morphism φ, produces a gluing rule: for each $x \in X_m$ it picks up some n-simplex Δ_y^n, where $y = X(\varphi)x \in X_n$, that should be glued to the constructed space $|X|$ as the φ-th face of simplex Δ_x^m.

Exercise 1.6. Is there a triangulation of the cycle S^1 by a) three 0-simplexes and three 1-simplexes[3] b) one 0-simplex and one 1-simplex? Is there a triangulation of the 2-sphere S^2 by c) four 0-simplexes, six 1-simplexes and four 2-simplexes d) two 0-simplexes, one 1-simplex and one 2-simplexes? Is there a triangulation of the 2-torus T^2 by one 0-simplex, three 1-simplexes and two 2-simplex?

Example 1.8 (simplicial sets)
Presheaves $X : \Delta^{\mathrm{opp}} \to \mathcal{S}et$ on the whole of the simplicial category are called *simplicial sets*. Each simlicial set X also produces a topological space $|X|$ called a *geometric realization* of X. It is glued from disjoint regular simplexes Δ_x^n, $x \in X_n$, by identifying points $s \in \Delta^n_{\varphi^*(x)}$ and $\varphi_*(s) \in \Delta_x^m$, where $\varphi : [n] \to [m]$ is a morphism in category Δ, $\varphi^* \overset{\text{def}}{=} X(\varphi) : X_m \to X_n$ denotes its image under X, and $\varphi_* : \Delta^n \to \Delta^m$ denotes affine linear map whose action on the vertexes of Δ^n is prescribed by φ. Formally speaking, $|X|$ is a quotient space of a topological direct product[4] $\prod_{n\geqslant 0} X_n \times \Delta^n$ by the minimal equivalence relation that contains identifications $\big(x, \varphi_* s\big) \simeq \big(\varphi^* x, s\big)$ for all arrows $\varphi : [n] \to [m]$ in $\mathrm{Mor}(\Delta)$, all $x \in X_m$, and all $s \in \Delta^n$.

If an arrow $\varphi = \delta\sigma : [n] \to [m]$ is decomposed into a surjection $\sigma : [n] \twoheadrightarrow [k]$ followed by an injection $\delta : [k] \hookrightarrow [m]$, then n-simplex Δ_z^n marked by $z = \sigma^* y = \sigma^*\delta^* x \in \varphi^*(X_m) \subset X_n$ appears in the space $|X|$ as k-simplex Δ_y^k obtained from Δ^n by means of linear projection $\sigma_* : \Delta^n \twoheadrightarrow \Delta^k$ and this k-simplex has to be the δ-th face of m-simplex Δ_x^m. In particular, all simplexes $z \in X_n$

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[1] i.e. functions $f(x)$ with $f'(x) = 1/x$

[2] that is, strictly increasing

[3] i.e. can one get S^1 as the geometric realization of a semisimplicial set X whose X_0 and X_1 consist of 3 elements and all other X_k are empty?

[4] where sets X_n are considered with the *discrete* topology and topologies on simplexes $\Delta^n \subset \mathbb{R}^{n+1}$ are iduced by the standard topologies on $\mathbb{R}^{n+1}$

lying in the image of any map σ^* coming from an arrow $\sigma : [k] \to [n]$ with $k > n$ are *degenerated*: they are visible in the space $|X|$ as simplexes of a smaller dimension.

Usage of degenerated simplexes allows to describe combinatorially more complicated cell complexes than the triangulations. For example, topological description of n-spere S^n as a quotint space $S^n = \Delta^n / \partial \Delta^n$ leads to a pseudo-triangulation of S^n by one 0-simplex and one n-cell, which is the interior part of the regular n-simplex Δ^n. Combinatorially, this is the geometric realization of simplicial set X that consists of sets X_k obtained from the sets $\mathrm{Hom}_\Delta([k],[n])$ by gluing all non-surjective maps to one distiguished element. The map $\varphi^* : X_m \to X_k$ corresponding to an arrow $\varphi : [k] \to [m]$ is induced by the left composition with φ:

$$\mathrm{Hom}_\Delta([m],[n]) \to \mathrm{Hom}_\Delta([k],[n]) , \quad \zeta \mapsto \varphi\zeta .$$

Exercise 1.7. Compute cardinalities[1] of all sets X_k and check that maps $\varphi^* : X_m \to X_k$ are well defined and produce a functor $X : \Delta^{\mathrm{opp}} \to \mathcal{S}et$.

1.2.2 Hom-functors. Associated with an object $X \in \mathrm{Ob}\,\mathcal{C}$ in an arbitrary category $\mathcal{C}$ are a (covariant) functor $h^X : \mathcal{C} \to \mathcal{S}et$ that takes $Y \in \mathrm{Ob}\,\mathcal{C}$ to $h^X(Y) \overset{\text{def}}{=} \mathrm{Hom}(X,Y)$ and sends an arrow $\varphi : Y_1 \to Y_2$ to the map $\varphi_* : \mathrm{Hom}(X,Y_1) \to \mathrm{Hom}(X,Y_2)\ \psi \mapsto \varphi \circ \psi$, provided by the left composition with φ and a presheaf $h_X : \mathcal{C} \to \mathcal{S}et$ that takes $Y \in \mathrm{Ob}\,\mathcal{C}$ to $h_X(Y) \overset{\text{def}}{=} \mathrm{Hom}(Y,X)$ and sends an arrow $\varphi : Y_1 \to Y_2$ to the map $\varphi^* : \mathrm{Hom}(Y_2,X) \to \mathrm{Hom}(Y_1,X)\ \psi \mapsto \psi \circ \varphi$ provided by the right composition with φ.

For example, presheaf $h_{[n]} : \Delta_s^{\mathrm{opp}} \to \mathcal{S}et$ produces the standard triangulation of the regular n-simplex Δ^n: the sets of k-simplexes $h_{[n]}([k]) = \mathrm{Hom}([k],[m])$ of this triangulation are precisely the sets of k-dimensional faces of Δ^n. Presheaf $h_U : \mathcal{U}(X) \to \mathcal{S}et$ on a topological space X has exactly one section over all open $W \subset U$ and the empty set of sections over all other open $W \not\subset U$. Presheaf $h_{\Bbbk} : \mathcal{V}ec_{\Bbbk}^{\mathrm{opp}} \to \mathcal{V}ec_{\Bbbk}$ takes a vector space V to its dual space $h_{\Bbbk}(V) = \mathrm{Hom}(V,\Bbbk) = V^*$ and sends a linear mapping $\varphi : V \to W$ to its dual mapping $\varphi^* : W^* \to V^*$, which takes a linear form $\xi : W \to \Bbbk$ to $\xi \circ \varphi : V \to \Bbbk$.

1.3 Natural transformations. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, then a *natural*[2] *transformation* is a collection of arrows $f_X : F(X) \to G(X)$, numbered by objects $X \in \mathrm{Ob}\,\mathcal{C}$, such that for each morphism $\varphi : X \to Y$ in $\mathcal{C}$ a diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ {\scriptstyle f_X}\big\downarrow & & \big\downarrow{\scriptstyle f_Y} \\ G(X) & \xrightarrow[G(\varphi)]{} & G(Y) \end{array} \tag{1-8}$$

is commutative in $\mathcal{D}$. A natural transformation $f : F \to G$ is called *an isomorphism of functors*, if all the morphisms $f_X : F(X) \to G(X)$ are isomorphisms. In this case functors F and G are called *isomorphic*.

On the language of algebras, a homomorphism $F : K[\mathcal{C}] \to K[\mathcal{D}]$ provides $K[\mathcal{D}]$ with a structure of a module over $K[\mathcal{C}]$, in which an element $a \in K[\mathcal{C}]$ acts on an element $b \in K[\mathcal{D}]$ as $a \cdot b \overset{\text{def}}{=} F(a) \cdot b$. Two functors F, G produce two different $K[\mathcal{C}]$-module structures on $K[\mathcal{D}]$ and

[1] note that $X_k \neq \varnothing$ for all $k \in \mathbb{Z}_{\geqslant 0}$

[2] or *functorial*

natural transformation $f : K[\mathcal{D}] \to K[\mathcal{D}]$ is nothing but a $K[\mathcal{C}]$-linear homomorphism between these modules: for each $\varphi \in K[\mathcal{C}]$ multiplications by $F(\varphi)$ and by $G(\varphi)$ in $K[\mathcal{D}]$ satisfy the relation $f \circ F(\varphi) = G(\varphi) \circ f$.

1.3.1 Categories of functors. If a category $\mathcal{C}$ is small, then the functors $\mathcal{C} \to \mathcal{D}$ to an arbitrary category $\mathcal{D}$ form a category $\mathcal{Fun}(\mathcal{C}, \mathcal{D})$, whose objects are the functors and morphismfs are the natural transformations. Contravariant functors $\mathcal{C}^{\text{opp}} \to \mathcal{D}$ also form a category called *a category of presheaves*[1] and denoted by $p\mathcal{Sh}(\mathcal{C}, \mathcal{D})$. Omitted letter $\mathcal{D}$ in this notation means on default that $\mathcal{D} = \mathcal{S}et$, i.e. $p\mathcal{Sh}(\mathcal{C}) \overset{\text{def}}{=} \mathcal{Fun}(\mathcal{C}^{\text{opp}}, \mathcal{S}et)$.

Exercise 1.8. Verify that prescription $X \mapsto h_X$ produces a covariant functor $\mathcal{C} \to p\mathcal{Sh}(\mathcal{C})$ and prescription $X \mapsto h^X$ produces a contravariant functor $\mathcal{C}^{\text{opp}} \to \mathcal{Fun}(\mathcal{C}, \mathcal{S}et)$.

1.3.2 Эквиваленности категорий. Categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent*, if there exists a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that compositions GF and FG are isomorphic to the identity functors $\text{Id}_{\mathcal{C}}$ and $\text{Id}_{\mathcal{D}}$ respectively. This does not mean that $FG = \text{Id}_{\mathcal{D}}$ or $GF = \text{Id}_{\mathcal{C}}$: objects $GF(X)$ and X may be different as well as objects $FG(Y)$ and Y. But there are functorial in $X \in \text{Ob}\,\mathcal{C}$ and $Y \in \text{Ob}\,\mathcal{D}$ isomorphisms

$$GF(X) \xrightarrow{\sim} X \quad \text{and} \quad FG(Y) \xrightarrow{\sim} Y\,. \tag{1-9}$$

In these case functors F and G are called *quasi-inverse equivalences* between categories $\mathcal{C}$ and $\mathcal{D}$.

Example 1.9 (choice of bases)
Write $vec_{\Bbbk}$ for the category of finite dimensional vector spaces over a field $\Bbbk$ and $\mathcal{C} \subset vec_{\Bbbk}$ for its small full subcategory formed by coordinate spaces $\Bbbk^n$, $n \geqslant 0$, where we put $\Bbbk^0 = \{0\}$. Let us fix some basis in each vector space $V \in \text{Ob}\, vec_{\Bbbk}$ or, equivalently, an isomorphism[2]

$$f_V : V \xrightarrow{\sim} \Bbbk^{\dim(V)}\,, \tag{1-10}$$

and for $V = \Bbbk^n$ put $f_{\Bbbk^n} = \text{Id}_{\Bbbk^n}$. Define a functor $F : vec \to \mathcal{C}$ by sending a space V to $\Bbbk^{\dim V}$ and an arrow $\varphi : V \to W$ to composition $F(\varphi) = f_W \circ \varphi \circ f_V^{-1}$, which can be viewed as the matrix of φ in the chosen bases of V and W. Let us show that F is an equivalence of categories quasi-inverse to the tautological full inclusion $G : \mathcal{C} \hookrightarrow vec$. By the construction of F there is an explicit equality of functors[3] $FG = \text{Id}_{\mathcal{C}}$. Reverse composition $GF : vec \to vec$ takes values in the small subcategory $\mathcal{C} \subset vec$ whose cardinality is non-compatible with cardinality vec at all. However, the isomorphisms (1-10) give a natural transformation $\text{Id}_{vec} \to GF$, because all the diagrams (1-8)

$$\begin{array}{ccc}
\text{Id}_{vec}(V) = V & \xrightarrow{\ \varphi = \text{Id}_{vec}(\varphi)\ } & W = \text{Id}_{vec}(W) \\
\Big\downarrow{\scriptstyle f_V} & & \Big\downarrow{\scriptstyle f_W} \\
GF(V) = \Bbbk^{\dim V} & \xrightarrow{\ GF(\varphi) = f_W \circ \varphi \circ f_V^{-1}\ } & \Bbbk^{\dim W} = GF(W)
\end{array}$$

are commutative by the construction of F. Thus, the identity functor Id_{vec} is naturally isomorphic to GF.

[1]of objects of the category $\mathcal{D}$ on the category $\mathcal{C}$
[2]that sends the fixed basis to the standard basis in $\Bbbk^n$
[3]non just a natural isomorphism

Exercise 1.9. Show that category of finite ordered sets $\varDelta_{\text{big}}$ is equivalent to its small simplicial subcategory $\varDelta \subset \varDelta_{\text{big}}$.

Lemma 1.1
Functor $G : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories iff it is full, faithful, and *essentially surjective* (the latter means that for each $Y \in \operatorname{Ob}\mathcal{D}$ there is some $X = X(Y) \in \operatorname{Ob}\mathcal{C}$ such that $G(X)$ is isomorphic to Y).

Proof. For each $Y \in \operatorname{Ob}\mathcal{D}$ pick up some $X = X(Y) \in \operatorname{Ob}\mathcal{C}$ and an isomorphism $f_Y : Y \simeq G(X)$. When $Y = G(X(Y))$ put $f_{G(X)} = \mathrm{Id}_{G(X)}$. Define a functor $F : \mathcal{D} \to \mathcal{C}$ by sending $Y \in \operatorname{Ob}\mathcal{D}$ to $F(Y) = X(Y)$ and arrow $\varphi : Y_1 \to Y_2$ to an arrow $\psi : X(Y_1) \to X(Y_2)$ such that $G(\psi) = f_{Y_2} \circ \varphi \circ f_{Y_1}^{-1}$ (since $G : \operatorname{Hom}(X_1, X_2) \simeq \operatorname{Hom}(G(X_1), G(X_2))$ is an isomorphism, such arrow ψ exists and is unique). By construction, $FG = \mathrm{Id}_{\mathcal{C}}$ and for each morphism $\varphi : Y_1 \to Y_2$ we have commutative diagram

$$\begin{array}{ccc} \mathrm{Id}_{\mathcal{D}}(Y_1) = Y_1 & \xrightarrow{\ \varphi\ } & Y_2 = \mathrm{Id}_{\mathcal{D}}(Y_2) \\ {\scriptstyle f_{Y_1}}\downarrow & & \downarrow{\scriptstyle f_{Y_2}} \\ GF(Y_1) = X_1 & \xrightarrow{\ GF(\varphi) = G(\psi)\ } & X_2 = GF(Y_2)\,. \end{array}$$

Thus, morphisms $f_Y : Y \simeq G(X) = GF(Y)$ give a natural isomorphism between $\mathrm{Id}_{\mathcal{D}}$ and GF. □

Exercise 1.10. Show that dualizing functor $h_{\Bbbk} : vec_{\Bbbk} \to vec_{\Bbbk}\,,\ V \mapsto V^*$, is quasi-inverse to itself and produces autoantiequivalence of the category of finite dimensional vector spaces.

1.4 Representable functors. A presheaf $F : \mathcal{C}^{\text{opp}} \to \mathcal{S}et$ is called *representable*, if it is naturally isomorphic to presheaf h_X for some $X \in \operatorname{Ob}\mathcal{C}$. In this case we say that object X *a represents* presheaf F. Dually, a covariant functor $F : \mathcal{C} \to \mathcal{S}et$ is called *corepresentable*, if it is naturally isomorphic to covariant functor h^X for some $X \in \operatorname{Ob}\mathcal{C}$. In this case we say that object X *a corepresents* functor F.

Lemma 1.2 (contravariant Yoneda lemma)
For any presheaf of sets $F : \mathcal{C}^{\text{opp}} \to \mathcal{S}et$ on an arbitrary category $\mathcal{C}$ there is functorial in $F \in pSh(\mathcal{C})$ and in $A \in \mathcal{C}$ bijection $F(A) \simeq \operatorname{Hom}_{pSh(\mathcal{C})}(h_A, F)$. It takes an element $a \in F(A)$ to a natural transformation

$$f_X : \operatorname{Hom}(X, A) \to F(X)\,, \tag{1-11}$$

that sends an arrow $\varphi : X \to A$ to the image of element a under map $F(\varphi) : F(A) \to F(X)$. The inverse bijection takes a natural transformation (1-11) to the image of the identity $\mathrm{Id}_A \in h_A(A)$ under the map $f_A : h_A(A) \to F(A)$.

Proof. For any natural transformation (1-11), for any object $X \in \operatorname{Ob}\mathcal{C}$, and for any arrow $\varphi : X \to A$ commutative diagram (1-8)

$$\begin{array}{ccc} h_A(A) = \operatorname{Hom}(A, A) & \xrightarrow{\ h_A(\varphi)\ } & \operatorname{Hom}(X, A) = h_A(X) \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle f_X} \\ F(A) & \xrightarrow{\ F(\varphi)\ } & F(X)\,, \end{array} \tag{1-12}$$

forces the equality $f_X(\varphi) = F(\varphi)\big(f_A(\mathrm{Id}_A)\big)$, because the upper arrow in (1-12) sends Id_A to φ. Thus the whole of transformation $f : h_A \to F$ is uniquely recovered as soon the element $a = f_A(\mathrm{Id}_A) \in F(A)$ is given. Choosing some $a \in F(A)$ we obtain transformation (1-11) that sends $\varphi \in \mathrm{Hom}(X, A)$ to $f_X(\varphi) = F(\varphi)(a) \in F(X)$. It is natural, because for any arrow $\psi : Y \to X$ and any $\varphi \in h_A(X)$ we have $f_Y\left(h_A(\psi)\varphi\right) = f_Y(\varphi\psi) = F(\varphi\psi)a = F(\psi)F(\varphi)a = F(\psi)\left(f_X(\varphi)\right)$, i.e. $f_Y \circ h_A(\psi) = F(\psi) \circ f_X$ are the same maps $h_A(X) \to F(Y)$. □

Exercise 1.11 (covariant Yoneda lemma). For any covariant functor $F : \mathcal{C} \to \mathcal{S}et$ construct functorial in F and in $A \in \mathrm{Ob}\,\mathcal{C}$ bijection $F(A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}un(\mathcal{C},\mathcal{S}et)}(h^A, F)$.

Corollary 1.1
Covariant functor $X \mapsto h_X$ and contravariant functor $X \mapsto h^X$ are full and faithful. In other words, there are functorial in $A, B \in \mathrm{Ob}\,\mathcal{C}$ isomorphisms $\mathrm{Hom}_{p\mathcal{S}h(\mathcal{C})}(h_A, h_B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{F}un(\mathcal{C})}(h^A, h^B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Proof. Apply Yoneda lemmas to $F = h_B$ and $F = h^B$. □

Corollary 1.2
If a functor $F : \mathcal{C} \to \mathcal{S}et$ is (co)representable, then its (co)representing object is unique up to natural isomorphism.

Proof. If $F \simeq h^A \simeq h^B$ (or $F \simeq h_A \simeq h_B$), then the natural isomorphism between functors h_A and h_B (resp. between h^A and h^B) produces by cor. 1.1 an isomorphism between A and B in $\mathcal{C}$. □

1.4.1 Definitions via «universal properties».

The Yoneda lemmas provide us with two dual ways for transferring set-theoretical constructions from category $\mathcal{S}et$ to an arbitrary category $\mathcal{C}$. Namely, to define some set-theoretical operation on objects $X_i \in \mathrm{Ob}\,\mathcal{C}$, consider a presheaf $\mathcal{C}^{\mathrm{opp}} \to \mathcal{S}et$ that takes an object $Y \in \mathrm{Ob}\,\mathcal{C}$ to the set obtained from the sets $\mathrm{Hom}(Y, X_i)$ by the operation in question. If this presheaf is representable, we declare its representing object to be the result of our operation applied to the objects X_i. The dual way uses covariant in Y functors $\mathrm{Hom}(X_i, Y)$ and corerepresentig object. Although both definitions are implicit, defined objects (if exist) come with some universal properties and are unique up to unique isomorphism respecting these properties.

Example 1.10 (direct product $A \times B$)
A *product* $A \times B$ of objects $A, B \in \mathrm{Ob}\,\mathcal{C}$ in an arbitrary category $\mathcal{C}$ is defined as representing object for presheaf of sets $Y \mapsto \mathrm{Hom}(Y, A) \times \mathrm{Hom}(Y, B)$. If $A \times B$ exists, then for all Y in $\mathcal{C}$ there is functorial in Y isomorphism $\beta_Y : \mathrm{Hom}(Y, A \times B) \xrightarrow{\sim} \mathrm{Hom}(Y, A) \times \mathrm{Hom}(Y, B)$. For $Y = A \times B$ it produces a pair of arrows $A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$ — the image of the identity $\beta_{A\times B}(\mathrm{Id}_{A\times B}) \in \mathrm{Hom}(A \times B, A) \times \mathrm{Hom}(A \times B, B)$. This pair is *universal* in the following sense: for any pair of arrows $A \xleftarrow{\varphi} Y \xrightarrow{\psi} B$ there exists a unique arrow $\varphi \times \psi : Y \to A \times B$ such that $\varphi = \pi_A \circ (\varphi \times \psi)$ and $\psi = \pi_B \circ (\varphi \times \psi)$.

Exercise 1.12. Show that a) for each diagram $A \xleftarrow{\pi'_A} C \xrightarrow{\pi'_B} B$ that possess the same universal property there exists a unique isomorphism $\gamma : C \xrightarrow{\sim} A \times B$ such that $\pi_A \circ \gamma = \pi'_A$ and $\pi_B \circ \gamma = \pi'_B$ b) for any pair of arrows $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ there is a unique arrow $\alpha \times \beta : A_1 \times B_1 \to A_2 \times B_2$ such that $\alpha \circ \pi_A = (\alpha \times \beta) \circ \alpha$ and $\beta \circ \pi_B = (\alpha \times \beta) \circ \beta$.

Exercise 1.13. Show that the product in $\mathcal{Top}$ exists and coincides with the set theoretical product $A \times B = \{(a,b) \mid a \in A,\ b \in B\}$ equipped with the weakest topology in which both maps π_A, π_B are continuous. Being equipped with componentwise operations, the set $A \times B$ turns to direct product in the categories of groups, rings and modules over a ring.

Example 1.11 (direct coproduct $A \otimes B$)
Dually, a *coproduct* $A \otimes B$ in an arbitrary category $\mathcal{C}$ is defined as corepresenting object for covariant functor $\mathcal{C} \to \mathcal{S}et$, $Y \mapsto \mathrm{Hom}(A, Y) \times \mathrm{Hom}(B, Y)$. It is uniquely characterized by the universal diagram $A \xrightarrow{\iota_A} A \otimes B \xleftarrow{\iota_B} B$ such that for any pair of arrows $A \xrightarrow{\varphi} Y \xleftarrow{\psi} B$ there exists a unique arrow $\varphi \otimes \psi : A \otimes B \to Y$ such that $\varphi = (\varphi \otimes \psi) \circ \iota_A$ and $\psi = (\varphi \otimes \psi) \circ \iota_B$.

Exercise 1.14. Let universal diagram $A \xrightarrow{\iota_A} A \otimes B \xleftarrow{\iota_B} B$ exist. Show that a) it is unique up to unique isomorphism commuting with ι_A and ι_B b) each pair of arrows $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ produces a unique arrow $\alpha \otimes \beta : A_1 \otimes B_1 \to A_2 \otimes B_2$ such that $\iota_A \circ \alpha = (\alpha \otimes \beta) \circ \alpha$.

In $\mathcal{S}et$ and $\mathcal{Top}$ the coproduct $A \otimes B = A \sqcup B$ is the disjoint union. In $\mathcal{G}rp$ the coproduct $A \otimes B = A * B$ is the free product[1]. In category of modules over a ring[2] $A \otimes B = A \times B = A \oplus B$ is the direct sum of modules. In the category of commutative rings with unity $A \otimes B$ is the tensor product of rings[3].

[1] i.e. the quotient of free group generated by $(A \setminus e) \sqcup (B \setminus e)$ through the minimal normal subgroup of relations that allow to replace any pair of consequent elements of the same group by their product in that group; for example, $\mathbb{Z} * \mathbb{Z} \simeq \mathbb{F}_2$ is free (non-commutative) group on two generators

[2] in particular, in $\mathcal{Ab}$

[3] It coincides with the tensor product of underlying abelian groups in the category of $\mathbb{Z}$-modules. The multiplication is defined as $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) \stackrel{\text{def}}{=} (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$

Comments to some exercises

Exrc. 1.3. Typical answer «$\ln|x| + C$, where C is an arbitrary constant» is *incorrect*. Actually, C is a section of the constant sheaf $\mathbb{R}^{\sim}$ over $\mathbb{R} \smallsetminus \{0\}$.

Exrc. 1.11. Each natural transformation f_* picks up an element in $F(A)$ — the image of the identity $\mathrm{Id}_A \in h^A(A)$ under the map $f_A : h^A(A) \to F(A)$. Vice versa, an element $a \in F(A)$ produces a transformation $f_X : \mathrm{Hom}(A, X) \to F(X)$ that sends an arrow $\varphi : A \to X$ to the image of a under the map $F(\varphi) : F(A) \to F(X)$. To verify that it is natural and takes $\mathrm{Id}_A \in h^A(A)$ to a via $f_A : h^A(A) \to F(A)$, use commutative diagram

$$\begin{array}{ccc} h^A(A) = \mathrm{Hom}(A,A) & \xrightarrow{h^A(\varphi)} & \mathrm{Hom}(A,X) = h^A(X) \\ \big\downarrow{\scriptstyle f_A} & & \big\downarrow{\scriptstyle f_X} \\ F(A) & \xrightarrow{F(\varphi)} & F(X)\,, \end{array} \qquad (1\text{-}13)$$

whose upper arrow sends Id_A to φ and forces $f_X(\varphi) = F(\varphi)\big(f_A(\mathrm{Id}_A)\big)$.