Categories and Functors

Notations. Let $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{A}b$, $\mathcal{G}rp$, $\mathcal{C}mr$, $\mathcal{M}od_K$, $\mathcal{V}ec_{\Bbbk} = \mathcal{M}od_{\Bbbk}$, $\mathcal{A}ss_{\Bbbk}$, A-$\mathcal{M}od$, $\mathcal{M}od$-A denote, respectively, the categories of sets, topological spaces, abelian groups, all groups, commutative rings[1], modules over a commutative ring K, vector spaces and associative algebras over a field $\Bbbk$, left and right modules over a (noncommutative) associative $\Bbbk$-algebra A. The categories of functors $\mathcal{C} \to \mathcal{D}$ and presheaves $\mathcal{C}^{\text{opp}} \to \mathcal{D}$ are denoted by $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ and $p\mathcal{S}h(\mathcal{C}, \mathcal{D})$.

SHA1⋄1. Let $\varDelta_{\text{big}}$ be the category of all finite ordered sets with order preserving maps as the morphisms and $\varDelta \subset \varDelta_{\text{big}}$ be its full small subcategory formed by sets $[n] \overset{\text{def}}{=} \{0, 1, \dots, n\}$, $n \geqslant 0$, ordered usually. Show that **a)** $\varDelta$ and $\varDelta_{\text{big}}$ are equivalent **b)** algebra $\mathbb{Z}[\varDelta]$ is generated by the identity arrows $e_n = \text{Id}_{[n]}$, the inclusions $\partial_n^{(i)} : [n-1] \hookrightarrow [n]$, $0 \leqslant i \leqslant n$, $i \notin \partial_n^{(i)}([n-1])$, and surjections $s_n^{(i)} : [n] \twoheadrightarrow [n-1]$, $0 \leqslant i \leqslant n-1$, $(i+1) \mapsto i$. **c*)** Find generators for the ideal of relations between these generating arrows.

SHA1⋄2. For a given $X \in \text{Ob}\,\mathcal{C}$, define the functor $h^X \mapsto \text{Hom}(X, Y)$ and the presheaf $h_X : Y \mapsto \text{Hom}(Y, X)$ by sending an arrow $\varphi : Y_1 \to Y_2$ to the maps

$$\varphi_* : \text{Hom}(X, Y_1) \to \text{Hom}(X, Y_2),\ \psi \mapsto \varphi \circ \psi,$$
$$\varphi^* : \text{Hom}(Y_2, X) \to \text{Hom}(Y_1, X),\ \psi \mapsto \psi \circ \varphi,$$

provided by the left and right multiplications by φ. Show that the assignments $X \mapsto h^X$ and $X \mapsto h_X$ define a pre-sheaf $h^* : \mathcal{C}^{\text{opp}} \to \mathcal{F}un(\mathcal{C}, \mathcal{S}et)$ and a functor $h_* : \mathcal{C} \to p\mathcal{S}h(\mathcal{C}, \mathcal{S}et)$ respectively.

SHA1⋄3. Show that functor $h^X : \mathcal{A}b \to \mathcal{A}b$ takes an exact sequence $0 \to A \to B \to C \to 0$ to the exact sequence $0 \to \text{Hom}(X, A) \to \text{Hom}(X, B) \to \text{Hom}(X, C)$ whose rightmost arrow may be non-surjective. Formulate and prove dual property of functor $h_X : \mathcal{A}b \to \mathcal{A}b$.

SHA1⋄4. Describe products and coproducts in **a)** $\mathcal{S}et$ **b)** $\mathcal{T}op$ **c)** $\mathcal{M}od_K$ **d)** $\mathcal{G}rp$ **e)** $\mathcal{C}mr$.

SHA1⋄5. Fix prime $p \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $A_n = \mathbb{Z}/(p^n)$. For $m > n$, write $\psi_{nm} : A_m \twoheadrightarrow A_n$ for the quotient map and $\varphi_{mn} : A_n \hookrightarrow A_m$ for the embedding $[1] \mapsto [p^{m-n}]$ respectively. In category $\mathcal{A}b$ describe **a)** $\lim A_n$ along the arrows ψ_{mn} **b)** $\text{colim}\, A_n$ along the arrows φ_{mn}.

SHA1⋄6. Let $B_n = \mathbb{Z}/(n)$. For $n|m$, write $\psi_{nm} : B_m \twoheadrightarrow B_n$ and $\varphi_{mn} : B_n \hookrightarrow B_m$ for the quotient map and the embedding $[1] \mapsto [m/n]$ respectively. In category $\mathcal{A}b$ describe **a)** $\lim B_n$ along the arrows ψ_{nm} **b)** $\text{colim}\, B_n$ along the arrows φ_{mn}.

SHA1⋄7. Prove that a functor $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint functor F iff for each $X \in \text{Ob}\,\mathcal{C}$, the functor $h_G^X : Y \mapsto \text{Hom}_{\mathcal{C}}(X, G(Y))$ is corepresentable, and in this case, $F(X)$ corepresents h_G^X. Formulate and prove the dual criteria for the existence of the right adjoint functor G to a given functor $F : \mathcal{C} \to \mathcal{D}$.

SHA1⋄8. Show that each left adjoint functor commutes with colimits and each right adjoint functor commutes with limits[2].

SHA1⋄9. For an arbitrary extension $S \subset R$ of associative algebras with units construct left and right adjoint functors to the restriction functor $\text{res}_S^R : R\text{-}\mathcal{M}od \to S\text{-}\mathcal{M}od$.

SHA1⋄10. Given a topological space X, write $S(X) : \varDelta^{\text{opp}} \to \mathcal{S}et$ for the simplicial set that takes $[n] \in \text{Ob}\,\varDelta$ to $S_n(X) \overset{\text{def}}{=} \text{Hom}_{\mathcal{T}op}(\varDelta^n, X)$, where $\varDelta^n \subset \mathbb{R}^{n+1}$ is the standard regular n-dimensional simplex, and takes an order preserving map $\varphi : [n] \to [m]$ to the map $f \mapsto f \circ |\varphi|$ provided by the right composition with the affine map $|\varphi| : \varDelta^n \to \varDelta^m$ acting on the vertices as φ. Show that the functor $S : \mathcal{T}op \to p\mathcal{S}h(\varDelta)$ is right adjoint to the geometric realization functor $p\mathcal{S}h(\varDelta) \to \mathcal{T}op$.

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[1] With the unit elements and the homomorphisms respecting the unit elements.

[2] A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *commuting with (co) limits*, if for every $L \in \text{Ob}\,\mathcal{C}$ and diagram $\Phi : \mathcal{N} \to \mathcal{C}$ the condition «L is the (co) limit of Φ in $\mathcal{C}$» implies the condition «$F(L)$ is the (co) limit of $F \circ \Phi$ in $\mathcal{D}$

Individual report card of ______________________________________. Task 1 (11.01.2018)

(write your name and surname)

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