Exact Categories[1]

SHA 2½◇1. Let a category $\mathcal{E}$ have the zero object 0 and the kernel and cokernel of every morphism[2]. Given a morphism $\varphi : X \to Y$ in $\mathcal{E}$, put $\operatorname{im}\varphi \overset{\text{def}}{=} \ker\big(Y \to \operatorname{coker}\varphi\big)$ and $\operatorname{coim}\varphi \overset{\text{def}}{=} \operatorname{coker}\big(\ker\varphi \to X\big)$. Show that every φ admits the functorial in φ factorization $X \to \operatorname{coim}\varphi \to \operatorname{im}\varphi \to Y$.

Exact categories. A category $\mathcal{E}$ is called to be *exact* if it satisfies the conditions of prb. SHA 2½◇1 an the canonical arrow $\operatorname{coim}\varphi \to \operatorname{im}\varphi$ is an isomorphism for every $\varphi \in \operatorname{Mor}\mathcal{E}$. A composition $\varphi\psi$ is called to be *exact* if $\ker\varphi = \operatorname{im}\psi$. All the remaining problems deal with an *arbitrary* exact category.

SHA 2½◇2. Consider a commutative diagram with exact arrows

$$\begin{array}{ccccc} X_1 & \xrightarrow{\alpha_1} & Y_1 & \xrightarrow{\beta_1} & Z_1 \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\zeta} \\ X_2 & \xrightarrow[\alpha_2]{} & Y_2 & \xrightarrow[\beta_2]{} & Z_2 \end{array}$$

a) For $\ker\alpha_1 = 0 = \ker\alpha_2$, construct an exact sequence $0 \to \ker\xi \to \ker\eta \to \ker\zeta$. Now assume that $\operatorname{coker}\xi = \operatorname{coker}\beta_1 = \operatorname{coker}\beta_2 = 0$. Show that **b)** $\operatorname{coker}\eta \simeq \operatorname{coker}\zeta$ **c)** the sequence $X_2 \to \operatorname{im}\eta \to \operatorname{im}\zeta$ is exact **d)** $\operatorname{im}\zeta = \operatorname{coker}\big(\ker\eta \to Z_1\big)$ **e)** $\operatorname{coker}\big(\ker\eta \to \ker\zeta\big) = 0$.

SHA 2½◇3. Given a commutative diagram with exact arrows

$$\begin{array}{ccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & D_2 \end{array}$$

show that if $\operatorname{coker}\alpha = 0$ (resp. $\ker\delta = 0$), then there exists an exact sequence $\ker\beta \to \ker\gamma \to \ker\delta$ (resp. $\operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$).

SHA 2½◇4. For every composition $\varphi\psi$, construct a long exact sequence $0 \to \ker\psi \to \ker\varphi\psi \to \ker\varphi \to \operatorname{coker}\psi \to \operatorname{coker}\varphi\psi \to \operatorname{coker}\varphi \to 0$.

SHA 2½◇5. For a commutative diagram with exact arrows

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 & \longrightarrow & E_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & D_2 & \longrightarrow & E_2 \end{array} \tag{1}$$

and $\operatorname{coker}\alpha = 0 = \ker\varepsilon$, put $K \overset{\text{def}}{=} \ker(C_1 \to D_2)$, $\overline{K} \overset{\text{def}}{=} \operatorname{coker}(B_1 \to C_2)$. **a)** Construct an injection $K \twoheadrightarrow \ker\delta$ and a surjection $\operatorname{coker}\beta \hookrightarrow \overline{K}$. **b)** Show that there exists a unique morphism $\partial : \ker\delta \to \operatorname{coker}\beta$ such that the compositions $K \to C_1 \to C_2 \to \overline{K}$ and $K \to \ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \to \overline{K}$ coincide. **c)** Construct an exact sequence $\ker\beta \to \ker\gamma \to \ker\delta \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to \operatorname{coker}\delta$.

SHA 2½◇6. For a diagram (1) with invertible α, β, δ, ε, show that γ is invertible too.

SHA 2½◇7. Given a complex[3] C: $\cdots \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots$, put $Z^i \overset{\text{def}}{=} \ker d^i$, $\overline{Z}^i \overset{\text{def}}{=} \operatorname{coker} d^{i-1}$, $H^i \overset{\text{def}}{=} Z^i / \operatorname{im} d^{i-1}$. Verify that d^n gives a morphism $\overline{Z}^n \to Z^{n+1}$. For an exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ construct the functorial exact sequence $H^n(A) \xrightarrow{H^n(\alpha)} H^n(B) \xrightarrow{H^n(\beta)} H^n(C) \xrightarrow{H^n(\delta^n)} H^{n+1}(A) \xrightarrow{H^{n+1}(\alpha)} H^{n+1}(B) \xrightarrow{H^{n+1}(\beta)} H^{n+1}(B)$.

[1] Hints for all the problems in this task see in *B. Iversen*. «Cohomology of sheaves».

[2] That is, the (co)equalizer of the morphism in question and the *zero morphism* (i.e., transmitted through the zero object).

[3] This means that $d^i d^{i-1} = 0$ for all i.

Individual report card of ______________________________________.
(write your name and surname)

Task № $2\frac{1}{2}$ (optional)

№	date	verified by	signature
1			
2a			
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