Adjoint Functors, Exact Functors, and Colimits.

SHA2◇1. Presheaves $F : \mathcal{C}^{\text{opp}} \to \mathcal{D}$, $G : \mathcal{D}^{\text{opp}} \to \mathcal{C}$ are called *left* (resp. *right*) *adjoint*, if there exist a natural in $C \in \operatorname{Ob}\mathcal{C}$, $D \in \operatorname{Ob}\mathcal{D}$ bijection $\operatorname{Hom}_{\mathcal{C}}\big(G(D), C\big) \simeq \operatorname{Hom}_{\mathcal{D}}\big(F(C), D\big)$ (resp. $\operatorname{Hom}_{\mathcal{C}}\big(C, G(D)\big) \simeq$ $\simeq \operatorname{Hom}_{\mathcal{D}}\big(D, F(C)\big)$). Formulate and solve the analog of SHA1◇8 for such presheaves.

SHA2◇2. Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ iff there are natural transformations $t : F \circ G \to \mathrm{Id}_{\mathcal{D}}$, $s : \mathrm{Id}_{\mathcal{C}} \to G \circ F$ such that the compositions of natural transformations[1] $F \xrightarrow{F\circ s} FGF \xrightarrow{t\circ F} F$ and $G \xrightarrow{s\circ G} GFG \xrightarrow{G\circ t} G$ are equal to the identity transformations of the functors F and G.

SHA2◇3. Show that a category $\mathcal{C}$ is coclosed iff it has an initial object, a coproduct for every set of objects, and a coequalizer for every pair of morphisms sharing the same domain and codomain.

SHA2◇4. Let a subset S in an associative (but not necessary commutative) ring R with unit be *multiplicative*, i.e., $1 \in S$, $s, t \in S \Rightarrow st \in S$, and satisfy the following two *Ore conditions*:

$$\text{for all } \varrho \in R,\ s \in S \text{ there exists } \lambda \in R,\ t \in S \text{ such that } \lambda s = t\varrho \qquad (O_1)$$

$$\forall\, \varphi, \psi \in R, \text{ if } \exists\, s \in S \text{ such that } \phi s = \psi s, \text{ then } \exists\, t \in S \text{ such that } t\phi = t\psi\,. \qquad (O_2)$$

Consider S as a category with $\operatorname{Hom}_S(s,t) \stackrel{\text{def}}{=} \{\lambda \in R \mid \lambda s = t\}$, and let a functor $S \to \mathcal{M}od\text{-}R$ send an object $s \in S$ to the free rank 1 right R-module spanned by the basis vector denoted by $[s^{-1}]$, and an arrow $\lambda \in \operatorname{Hom}_S(s_1, s_2)$ to the homomorphism acting on this basis vector as $[s_1^{-1}] \mapsto [s_2^{-1}] \cdot \lambda$. Write $S^{-1}R$ for the colimit of this diagram. Show that it is formed by the classes of formal fractions $s^{-1}\varrho$ modulo the relation $s_1^{-1}\varrho_1 \sim s_2^{-1}\varrho_2$ meaning an existence of $\lambda_1, \lambda_2 \in R$ such that $\lambda_1 s_1 = \lambda_2 s_2 \in S$ and $\lambda_1\varrho_1 = \lambda_2\varrho_2$, and define a structure of assotiative ring with unit on $S^{-1}R$.

SHA2◇5 (exact functors). A functor $F : \mathcal{A}b \to \mathcal{A}b$ (resp. a presheaf $\mathcal{A}b^{\text{opp}} \to \mathcal{A}b$) is called *left exact* if it sends the kernels (resp. the cokernels) to the kernels. Dually, F is called *right exact* if it sends the cokernels (resp. the kernels) to the cokernels. F is called *exact*, if it is both left and right exact. Prove that: **a)** for every $N \in \operatorname{Ob}\mathcal{A}b$, the functor $X \mapsto X \otimes_{\mathbb{Z}} N$ is right exact, and for some N, it is not left exct **b)** for every small category $\mathcal{N}$, a the colimit functor $\operatorname{colim} : \mathcal{F}un(\mathcal{N}, \mathcal{A}b) \to \mathcal{A}b$ is exact[2].

SHA2◇6. Show that a sequence of sheaves $0 \to F \to G \to H \to 0$ on a topological space X is exact iff for every point $x \in X$, the sequence of fibers $0 \to F_x \to G_x \to H_x \to 0$ is exact in $\mathcal{A}b$. Give an example showing that this fails for exact sequences of presheaves.

SHA2◇7. Show that the tautological embedding $\mathcal{S}h(X) \hookrightarrow p\mathcal{S}h(X)$ is left but not right exact.

SHA2◇8. Show that the functor $\Gamma : \mathcal{T}op(X) \to p\mathcal{S}h(X)$, which takes a continuous map $E \to X$ to the sheaf of its local sections, is right adjoint to the functor $\mathcal{E} : p\mathcal{S}h(X) \to \mathcal{T}op(X)$, which takes a sheaf of sets F on X to its étale space $\mathcal{E}_F = \coprod_{x\in X} F_x$ equipped with the least topology in which the section $s : U \to \mathcal{E}_F$, $x \mapsto$ (class s in F_x) is continuous for every open $U \subset X$ and every $s \in F(U)$.

[1] Here $(F \circ s)_X \stackrel{\text{def}}{=} F(s_X)$, $(t \circ F)_X \stackrel{\text{def}}{=} t_{F(X)}$ etc.

[2] The (co)kernel of a natural transformation $f : X \to Y$ is formed by the (co)kernels of maps $f_\nu : X_\nu \to Y_\nu$, $\nu \in \operatorname{Ob}\mathcal{N}$. Show that the arrows $X(\mu \to \nu)$ and $Y(\mu \to \nu)$ give well defined maps between (co)kernels, and these maps form the diagrams $\ker f : \mathcal{N} \to \mathcal{A}b$ and $\operatorname{coker} f : \mathcal{N} \to \mathcal{A}b$.

Individual report card of ____. Task 2 (25.01.2018)

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