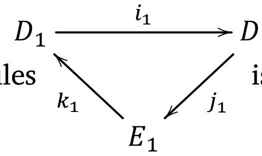
Spectral Sequences.

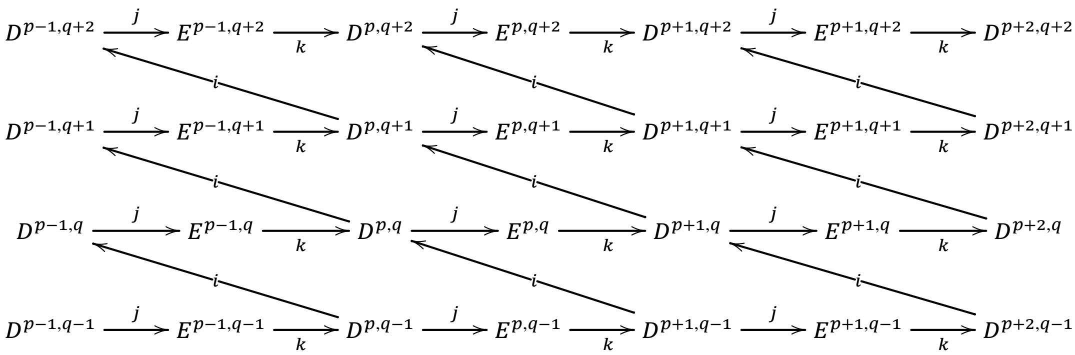
SHA4⋄1. An exact diagram of modules  is denoted by $(D_1, E_1, i_1, j_1, k_1)$ and called an *exact couple*. Put $d_1 = j_1 k_1$, $E_2 = \ker d_1 / \operatorname{im} d_1 \simeq k_1^{-1}(D_1)/j_1(\ker i_1)$, $D_2 = \operatorname{im} i_1$,

$$i_2 = i_1|_{\operatorname{im} i_1}, \quad j_2 : i_1(x) \mapsto j_1(x), \quad k_2 : x \,(\operatorname{mod} \operatorname{im} d_1) \mapsto k_1(x).$$

Show that **a)** $d_1^2 = 0$, j_2 and k_2 are well defined, and $(D_2, E_2, i_2, j_2, k_2)$ is an exact couple too (it is called the *derived couple* of $(D_1, E_1, i_1, j_1, k_1)$) **b)** In the $(r-1)$th derived couple $(D_r, E_r, i_r, j_r, k_r)$, we have $D_r \simeq \operatorname{im} i_1^{r-1}$, $E_r \simeq k_1^{-1}(\operatorname{im} i_1^{r-1})/j_1(\ker i_1^r)$ and the exact triple

$$0 \to \operatorname{im} i_1^{r-1}/\operatorname{im} i_1^r \to E_r \to \ker i_1^r / \ker i_1^{r-1} \to 0\,.$$

SHA4⋄2. Let modules $D_1 = \bigoplus_{p,q\in\mathbb{Z}} D_1^{p,q}$, $E_1 = \bigoplus_{p,q\in\mathbb{Z}} E_1^{p,q}$ be bigraded and equipped with the homogeneous morphisms of bidegrees $\deg i_1 = (-1, 1)$, $\deg j_1 = (0, 0)$, $\deg k_1 = (1, 0)$. Write $(D_r, E_r, i_r, j_r, k_r)$ for the $(r-1)$th derived couple of $(D_1, E_1, i_1, j_1, k_1)$ and put the modules $E_r^{p,q}$ in the cells of a rectangular table whose columns and rows are numbered by p and q respectively. Let $E_1^{p,q} = 0$ uniformly in p for $q \ll 0$ and uniformly in q for $p \ll 0$. For every cell (p, q), show that there exists $N = N(p, q) \in \mathbb{N}$ such that $\forall\, r > N$, $E_r^{p,q} = E_{r+1}^{p,q}$. Describe $E_\infty^{p,q}$ explicitly as a subfactor of D_1 in terms of kernels or images of the iterated maps $i_1 : D_1 \to D_1$ (see the diaram below).



Limit. Let $\forall\, p, q$, $\exists\, N = N(p, q)$ such that both incoming and outgoing differentials at the (p, q)-cell vanish in the table $E_r^{p,q}$ for all $r > N$. Then there are well defined modules $E_\infty^{p,q} \overset{\text{def}}{=} E_{N+1}^{p,q} = E_{N+2}^{p,q} = \dots$. If there exist some modules E_∞^n equipped with decreasing filtrations $F^p E_\infty^n$ such that $E_\infty^n = \bigcup_p F^p E_\infty^n$, $\bigcap_p F^p E_\infty^n = 0$ and $F^p E_\infty^n / F^{p+1} E_\infty^n = E_\infty^{p,n-p}$, then we say that $E_r^{p,q}$ are *converging* to E_∞^n and write $E_r^{p,q} \Rightarrow E_\infty^n$.

SHA4⋄3. Let every module K^m in a complex $\cdots \to K^m \to K^{m+1} \to \cdots$ be equipped with a finite decreasing filtration $K^m = F^0 K^m \supset F^1 K^m \supset F^2 K^m \supset \cdots \supset 0$ such that $d(F^p K^m) \subset F^p K^{m+1}$ for all p, m. Show that: **a)** for every p, there is a well defined quotient complex $G^p K$ whose degree m component is $F^p K^m / F^{p+1} K^m$ and the differential is induced the differential d in K **b)** the modules $D_1^{p,q} = H^{p+q}(F^p K)$ and $E_1^{p,q} = H^{p+q}(G^p K)$ form an exact couple whose rth derived couple has

$$E_{r+1}^{p,q} \simeq Z_r^{p,q} / \left(B_r^{p,q} \cap Z_r^{p,q}\right) \simeq \left(Z_r^{p,q} + B_r^{p,q}\right)/B_r^{p,q}\,,$$

where $Z_r^{p,q} \overset{\text{def}}{=} \{c \in F^p K^{p+q} \mid dc \in d\left(F^{p+r} K^{p+q}\right)\}$ and $B_r^{p,q} \overset{\text{def}}{=} d\left(F^{p-r} K^{p+q-1}\right) + F^{p+1} K^{p+q}$
c) $E_r^{p,q} \Rightarrow H^n(K)$.

SHA4⋄4. For the total complex $\operatorname{Tot}(K)$ of a bicomplex $K = \bigoplus K^{p,q}$, consider two filtrations ${}^{I}F$ and ${}^{II}F$ with ${}^{I}F^p \operatorname{Tot}^m(K) = \bigoplus_{\nu \geqslant p,\, p+q=n} K^{p,q}$ and ${}^{II}F^q \operatorname{Tot}^m(K) = \bigoplus_{\nu \geqslant q,\, p+q=n} K^{p,q}$. Construct two spectral sequences ${}^{I}E_r^{p,q} \Rightarrow H^n(\operatorname{Tot}(K))$ with ${}^{I}E_2^{p,q} = H_{d_1}^p(H_{d_2}^q(K))$ and ${}^{II}E_r^{p,q} \Rightarrow H^n(\operatorname{Tot}(K))$ with ${}^{II}E_2^{p,q} = H_{d_2}^q(H_{d_1}^p(K))$.

Individual report card of ____ .
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Task 4 (22.02.2018)

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