# Sections with a compact support.

**Conventions and notations.** In this Task, every (pre)sheaf is a (pre)sheaf of abelian groups, and all topological spaces are locally compact. A continuous map is called to be *proper* if the complete preimage of every compact is compact. A subset $W \subset X$ is called to be *locally closed* if every point $w \in W$ has an open neighborhood $U \ni w$ in $X$ such that $U \cap W$ is closed in the topology on $U$ induced from $X$. For a sheaf $F$ on $X$, the elements of abelian group

$$H^0_c(X,F) \stackrel{\text{def}}{=} \{s \in F(X) \mid \operatorname{supp}(s) \text{ is compact}\}$$

are called the *global sections with a compact support*.

**SHA6◇1.** For a continuous map $f : X \to Y$ and a sheaf $F$ on $X$ write $f_!F \subset f_*F$ for the presheaf on $Y$ with $f_!F(U) \stackrel{\text{def}}{=} \{s \in F(f^{-1}U) \mid \text{the map } f|_{\operatorname{supp}(s)} : \operatorname{supp}(s) \to U \text{ is proper}\}$. Show that: **a)** $f_!F$ is a sheaf **b)** if $f$ is the embedding of a closed subset, then $f_! = f_*$ **c)** the functor $f_! : \mathcal{S}h(X) \to \mathcal{S}h(Y)$, $F \mapsto f_!F$ is left exact **d)** $\forall\, y \in Y$ the stalk $f_!F_y \simeq H^0_c\left(f^{-1}(y), F|_{f^{-1}(y)}\right)$.

**SHA6◇2.** Let $f : W \hookrightarrow X$ be the embedding of a locally closed subset and $F$ a presheaf on $W$. Show that: **a)** the stalk $f_!F_x$ coincides with $F_x$ for $x \in W$, and vanishes for $x \notin W$ **b)** the functor $f_!$ is exact **c)** $f^*f_! = \mathrm{Id}_{\mathcal{S}h(W)}$ **d)** the functors $f_!$, $f^*$ are quasi-inverse equivalences between the category $\mathcal{S}h(W)$ and the full subcategory in $\mathcal{S}h(X)$ that consists of the sheaves with zero stalk at every point of $X \smallsetminus W$.

**SHA6◇3.** Under the conditions of prb. SHA6◇2, let $G$ be a sheaf on $X$. Put $G^W(U) \stackrel{\text{def}}{=} \{s \in F(U) \mid \operatorname{supp}(s) \subset W\}$ and $h^!G \stackrel{\text{def}}{=} h^*G^W$. Show that: **a)** $G^W$ is a sheaf on $X$ with zero stalk at every point of $X \smallsetminus W$ **b)** the functor $f^!$ is left exact and *right* adjoint to the functor $f_!$ **c)** if $W$ is open in $X$, then $h^! = h^*$ **d)** if $W$ is closed in $X$, then $h^!h_* = \mathrm{Id}_{\mathcal{S}h(W)}$.

**SHA6◇4.** Let $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ be the embeddings of some complementary closed subset $Z \subset X$ and open $U = X \smallsetminus Z$. Show that every sheaf $F$ on $X$ fits in the exact triple of sheaves $0 \to j_!j^*F \to F \to i_*i^*F \to 0$ on $X$.

**SHA6◇5.** For a continuous map $f : X \to Y$ and a localle closed embedding $h : W \hookrightarrow Y$, write $g$ for the locally closed embedding $g : f^{-1}(W) \hookrightarrow X$. Construct an isomorphism of functors $f_*h^! \simeq g_!f^*$.

**SHA6◇6.** For a continuous map $f : X \to Y$ and an exact triple of sheaves $0 \to F \to G \to H \to 0$ on $X$ with soft $F$, prove that: **a)** the sheaf $f_!F$ is soft and the sequences **b)** $0 \to f_!F \to f_!G \to f_!H \to 0$, of sheaves, **c)** $0 \to H^0_c(X,F) \to H^0_c(X,G) \to H^0_c(X,H) \to 0$, of abelian groups, are exact.

**SHA6◇7.** For a sheaf $F$ and $X$ and a continuous map $f : X \to Y$, the higher direct image with compact supports $R^q f_!F$ is defined as the $q$th cohomology sheaf of the complex of sheaves on $Y$

$$0 \to f_!C^0_F \to f_!C^1_F \to f_!C^2_F \to \cdots$$

obtained by applying $f_!$ to the flabby Godement's resolution $C^\bullet_F$ of the sheaf[1] $F$ on $X$. Show that: **a)** every exact triple of sheaves on $X$ produces the long exact sequence of higher direct images with compact supports on $Y$ **b)** in the definition of $R^q f_!F$, the Godements's resolution may be replaced by an arbitrary complex of sheaves $G^\bullet$ on $X$ such that the only nonzero cohomology sheaf of $G^\bullet$ is $H^0(G^\bullet) \simeq F$ and $R^q f_!G^p = 0$ for $q > 0$ and all $p$ **c)** there exists a spectral sequence[2] with $E_2^{p,q} = H^p_c(Y, R^q f_!F)$ converging to $H^{p+q}_c(X,F)$.

**SHA6◇8.** Write $\dim_c X$ for the minimal $n$ such that $H^n_c(X,F) = 0$ for every sheaf $F$ on $X$. Show that: **a)** given an exact sequence of sheaves $0 \to F \to S^0 \to S^1 \to \cdots \to S^{n-1} \to S^n \to 0$ such that $S^k$ is soft for $0 \leqslant k \leqslant n-1$, than $S^n$ is soft as well **b)** $\dim_c \mathbb{R}^n = n$ **c)** $\dim_c W \leqslant \dim_c X$ for every locally closed $W \subset X$ **d)** if every point $x \in X$ has an open neighborhood $U \ni x$ with $\dim_c U \leqslant n$, then $\dim_c X \leqslant n$ **e)** $R^q f_!F = 0$ for all $q > \dim_c X$, any continuous map $f : X \to Y$, and every sheaf $F$ on $X$.

[1] In particular, $H^q_c(X,F) = R^q c_!F$, where $c : X \to \mathrm{pt}$ is the constant map to a point.
[2] It is called the *Leray spectral sequence*.

Individual report card of ________________________________________ . Task 6 (22.03.2018)
(write your name and surname)

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