Proper support cohomology

Conventions and notations. In this Task, all sheaves are assumed to be the sheaves of abelian groups, and all topological spaces locally compact of finite cohomological dimension $\dim_c X < \infty$ (see prb. SHA6⋄8). The tensor product of sheaves F, G has the sections $F \otimes G(U) \overset{\text{def}}{=} L(U) \otimes F(U)$ (the tensor product of abelian groups) over open $U \subset X$. A sheaf L on X is called *flat* if the functor $F \mapsto L \otimes F$ is exact on the category of sheaves on X.

SHA8⋄1. Given a sheaf F on X and an embedding of arbitrary closed subset $\iota: Z \hookrightarrow X$, construct canonical isomorphism $\operatorname{colim}_{U \supset Z} H^0_c(U, F) \xrightarrow{\sim} H^0_c(Z, \iota^* F)$.

SHA8⋄2. Construct the Mayer–Vietoris long exact sequence as in prb. ΠΓA5⋄6 for the cohomologies with compact supports and a pair of arbitrary closed subsets $Z_1, Z_2 \subset X$.

SHA8⋄3. Given a sheaf F on X and an open subset $U \subset X$ with the complementary closed subset $Z = X \smallsetminus U$, construct long exact sequence $\cdots \to H^i_c(U, F) \to H^0_c(X, F) \to H^0_c(Z, F) \to H^{i+1}_c(U, F) \to \cdots$.

SHA8⋄4. Show that a sheaf F is soft iff $H^1_c(U, F) = 0$ for every open $U \subset X$.

SHA8⋄5. Compute the cohomologies with compact supports for the constant sheaves $\mathbb{R}$, $\mathbb{Z}$ on the space $\mathbb{R}^n$ and the half-space $x_1 \geqslant 0$ in $\mathbb{R}^n$.

SHA8⋄6. Construct a canonical isomorphism between the cohomologies with compact support of the constant sheaf $\mathbb{R}$ on a smooth manifold X and the cohomologies of De Rham complex of smooth global differential forms with compact support on X.

SHA8⋄7. For an open set $U \subset X$, write $\mathbb{Z}_U$ for the sheaf on X associated with the presheaf whose group of sections over an open W equals $\mathbb{Z}$ if $U \cap W \neq \varnothing$ and thero otherwise. Show that: **a)** the sheaf $\mathbb{Z}_U$ is flat and corepresents the functor $\Gamma_U: F \mapsto F(U)$ from sheaves to abelian groups **b)** every sheaf F on X is a cokernel of some map between direct sums of sheaves of the form $\mathbb{Z}_U$ **c)** every sheaf F on X is the colimit of naturally depending on F diagram of sheaves of the form $\mathbb{Z}_U$.

SHA8⋄8. Prove that a contravariant functor from the category of sheaves to the category of abelian groups is representable iff it sends colimits to limits.

SHA8⋄9. Prove that: **a)** the Godement resolution of a flat sheaf consists of flat soft sheaves **b)** every flat sheaf has a flat soft resolution of finite length.

SHA8⋄10. Given a flat soft sheaf L on X, prove that **a)** $L \otimes F$ is soft for every sheaf F **b)** for every continuous map $f: X \to Y$, the functor $\mathcal{S}h(X) \to \mathcal{S}h(Y)$, $F \mapsto f_!(L \otimes F)$ is exact and has the right adjoint, which sends injective sheaves to injective.

SHA8⋄11*. Let X be an orientable manifold of dimension n with boundary, $\omega_X = i_! \mathbb{R}$ be the sheaf on X obtained from the constant sheaf $\mathbb{R}$ on $X \smallsetminus \partial X$ via extension by zero.For every sheaf F on X construct canonical isomorphism $H^i_c(X, F)^* \simeq \operatorname{Ext}^{n-i}_{\mathcal{S}h(X)}(F, \omega_X)$.

Individual report card of ____. Task 8 (26.04.2018)
(write your name and surname)

№	date	verified by	signature
1			
2			
3			
4			
5			
6			
7a			
b			
c			
8			
9a			
b			
10a			
b			
11			